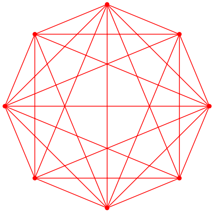
Subquadratic algorithms in minor-free digraphs:



(Weighted) distance oracles, decremental reachability, and more

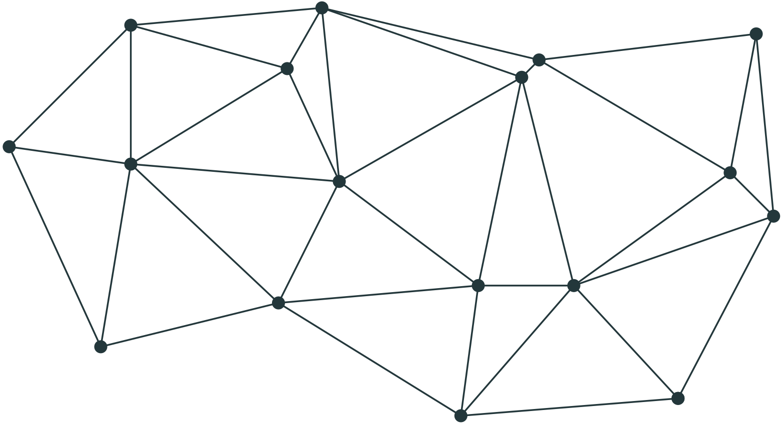
Adam Karczmarz (University of Warsaw & IDEAS NCBR) and **Da Wei Zheng** (UIUC)

Jan. 15, 2025

SODA 2025

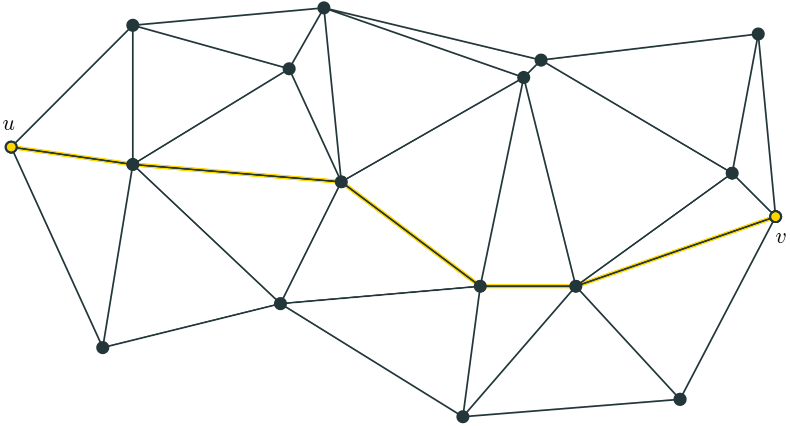
Two problems related to APSP

Diameter of a Graph



Diameter of a Graph

The **diameter** of a graph is the maximum length of a shortest path between $u, v \in V$.



Computing the Diameter

General graphs

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$O(mn)$ time by repeated BFS.

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Fine-grained reduction: if $O(m^{2-\varepsilon})$
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$\widetilde{O}(n^{2-1/3})$ time using *Voronoi diagrams*.
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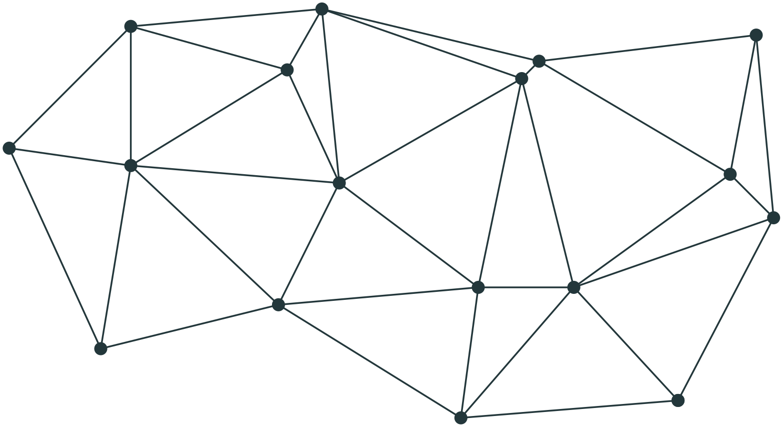
K_h**-minor-free graphs**

$\widetilde{O}(n^{2-1/(3h-1)})$ time using *VC dimension*.

Ducoffe, Habib, Viennot [SODA '20]

Le, Wulff-Nilsen [SODA '24]

Querying for distances: Compact APSP



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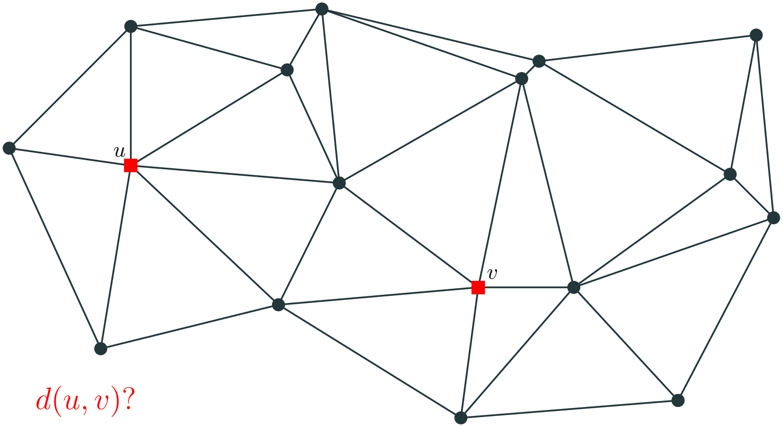

$d(u, v)$?

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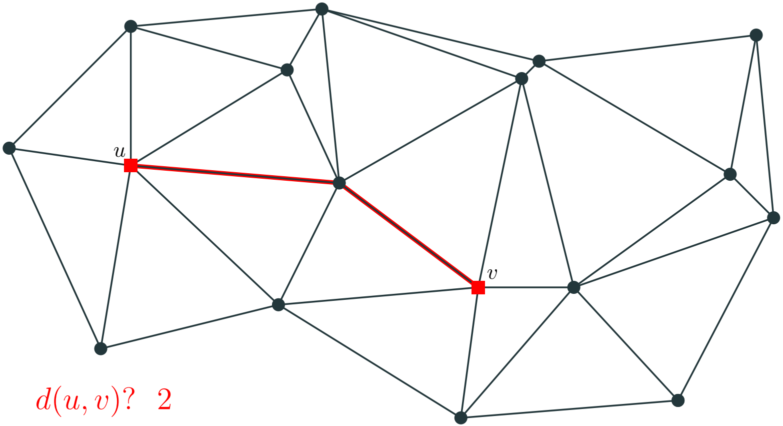

$d(u, v)$? 2

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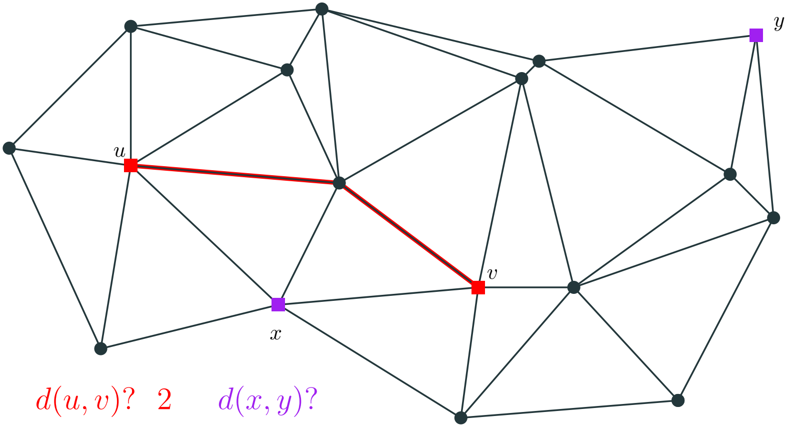

$d(u, v)$? 2 $\quad$ $d(x, y)$?

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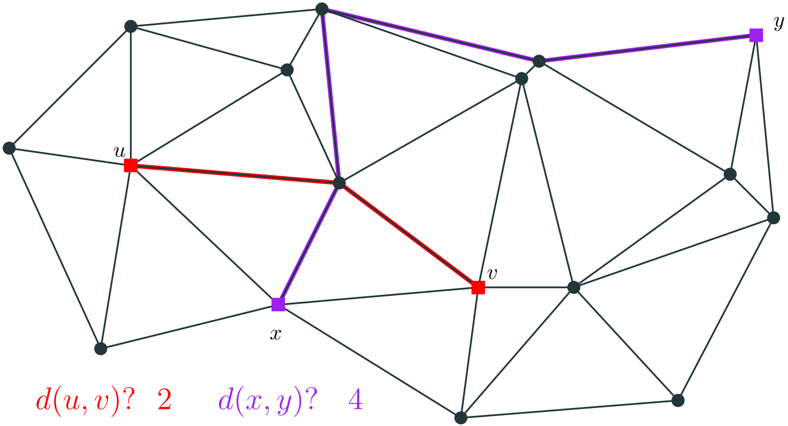

$d(u, v)?$ 2 $\quad d(x, y)?$ 4

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$O(n^{2-1/(4h-1)})$ space (real weights). **NEW!**

VC dimension

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Given a range space $(X, \mathcal{R})$, the VC dimension is size of maximum subset $S \subseteq X$ s.t:

$$\{(R \cap S) \mid R \in \mathcal{R}\} = 2^S$$

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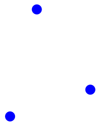

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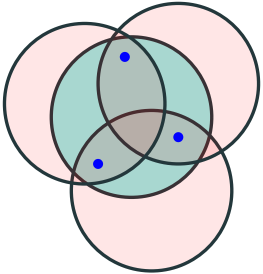

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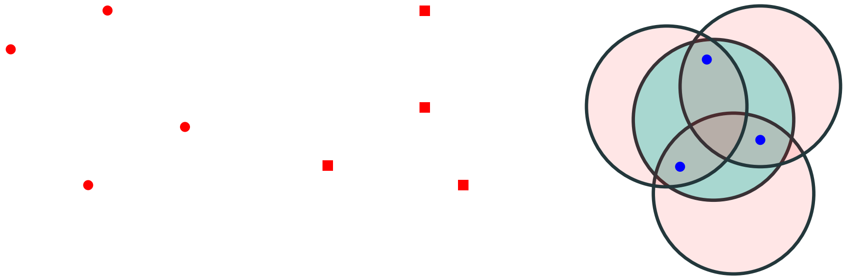

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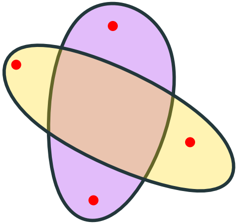

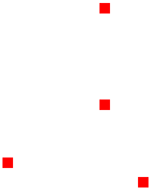

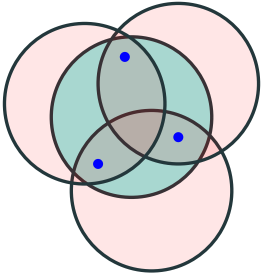

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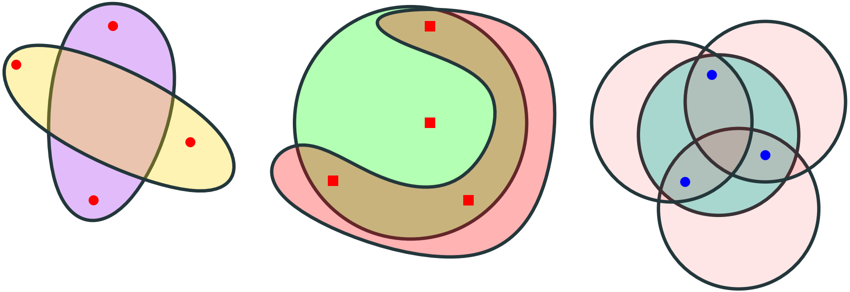
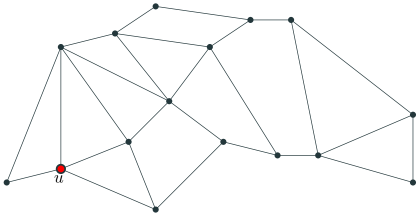


VC Dimension in Graphs

Definition: A ball $B(u, r)$ is the set of vertices distance $\leq r$ from $u \in V$.

Let $\mathcal{B}$ denote the set of all balls, i.e. $\mathcal{B} = \{B(u, r) \mid u \in V, r \in [n]\}$.



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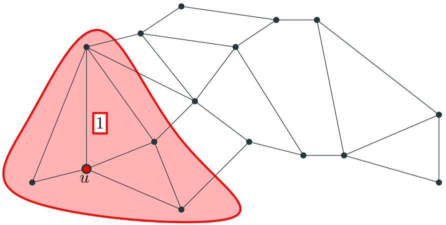

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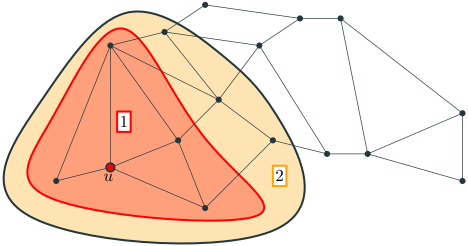

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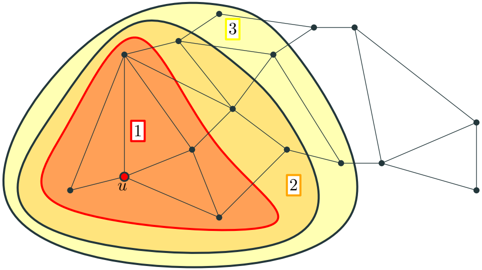

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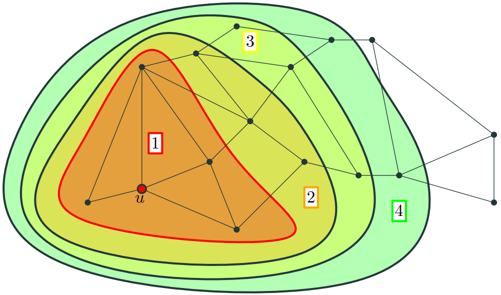

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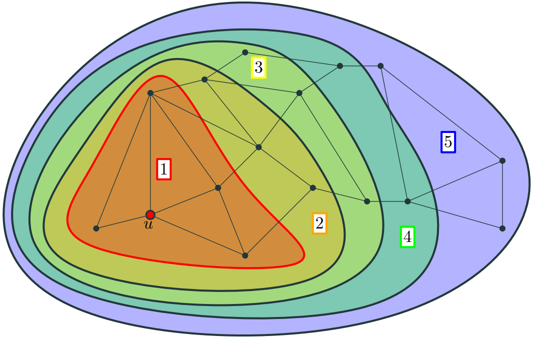

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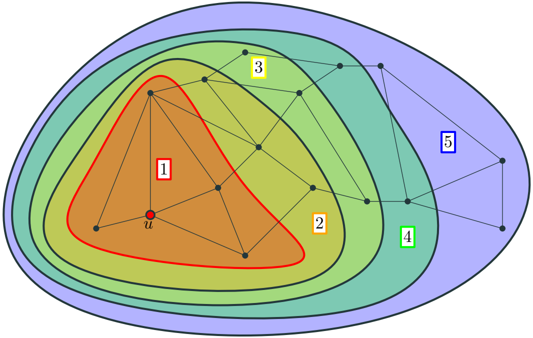
The <u>distance VC dimension</u> of G is $VCDim(\mathcal{B})$.



What graphs have bounded VC dimension?

Bounded distance VC dimension graphs

$\boxed{\textbf{Theorem.}\ \text{If } G \text{ is planar, then } VCDim(\mathcal{B}) \leq 4.}$

Chepoi, Estellon, Vaxès [DCG '07]

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Bousquet, Thomassé [Disc. Math. '15]

Bounded distance VC dimension graphs

Theorem. If G is planar, then $VCDim(\mathcal{B}) \leq 4$.

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Theorem. If G is K_h-minor-free, then $VCDim(\mathcal{B}) \leq h - 1$.

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Theorem. If G is a unit disk intersection graph, then $VCDim(\mathcal{B}) \leq 4$.

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Proof sketch of bounded VC dimension

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Example: K_4-minor free graph. Suppose distance VC dimension was 4. Minimal balls.

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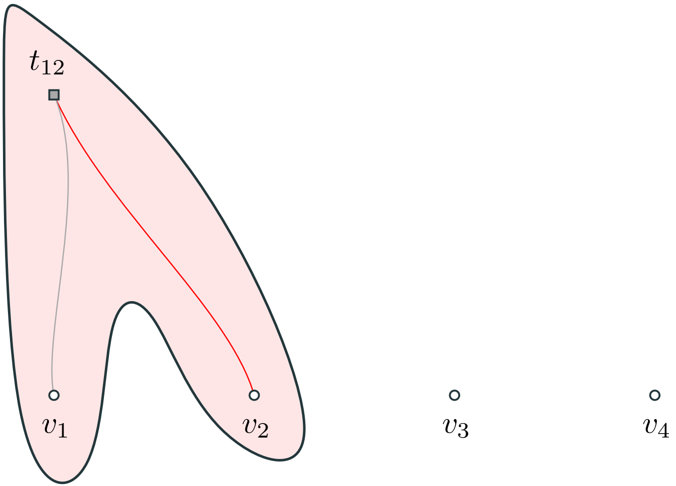

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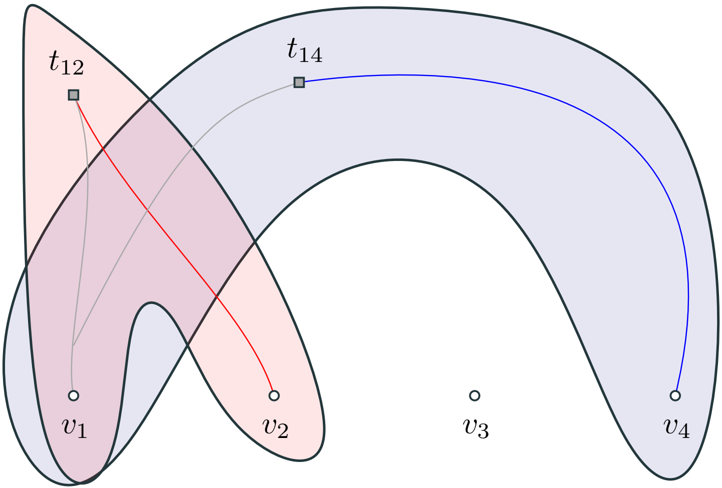

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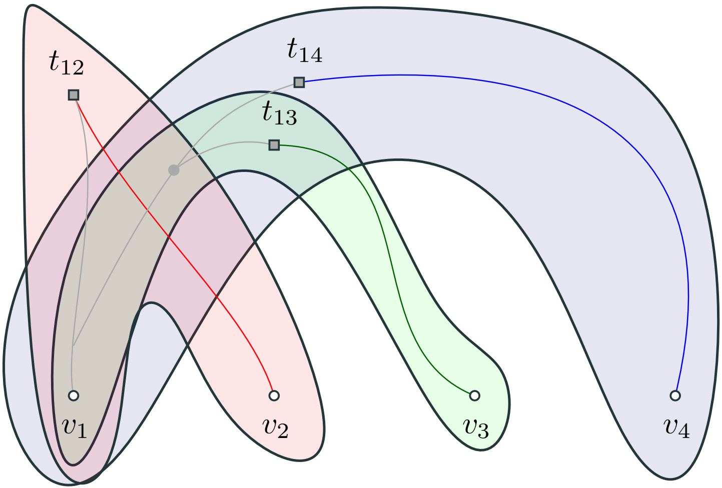

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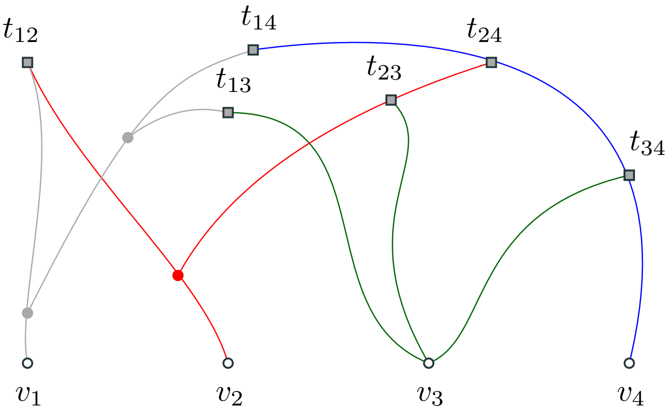

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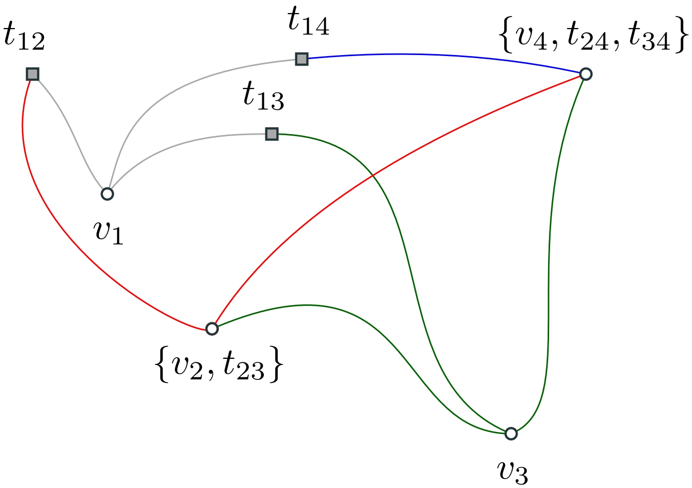
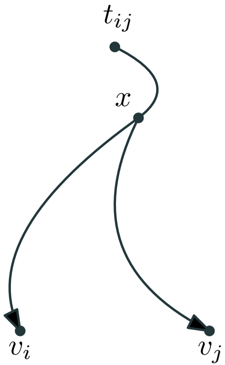


Non-crossing of ball paths

Case 1: Share both



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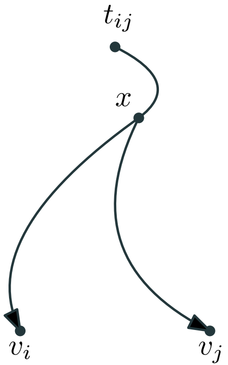
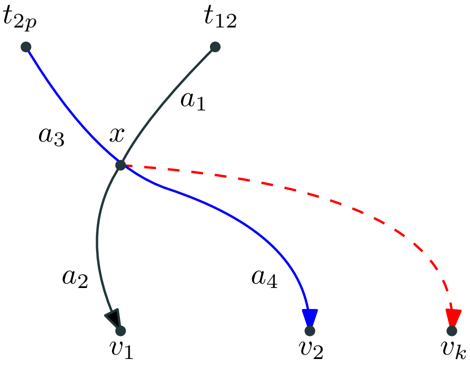


Case 2: Share one



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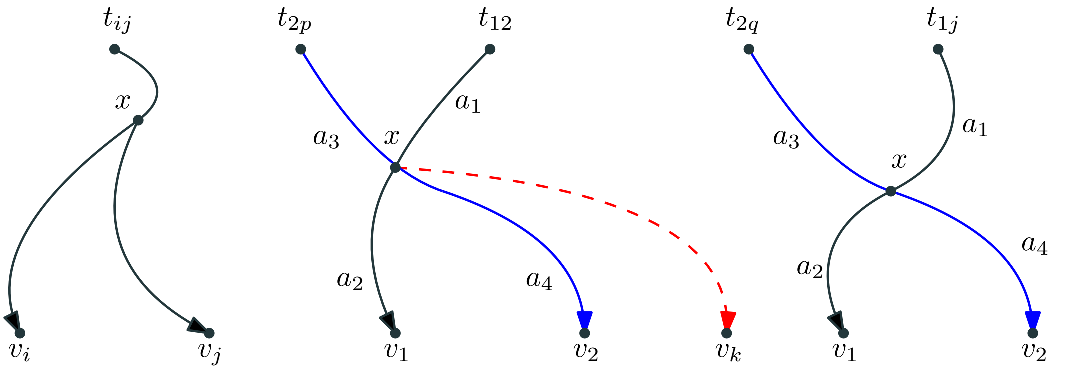

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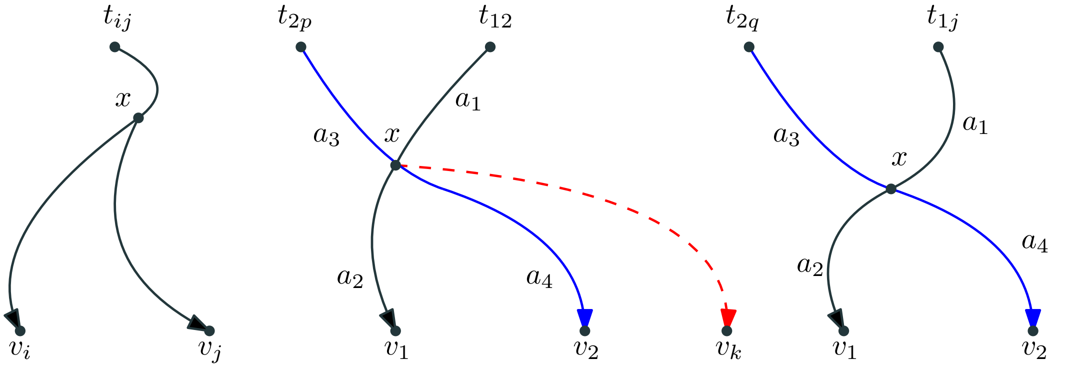

$$d(t_{1j}, v_1) + d(t_{2q}, v_2) = a_1 + a_2 + a_3 + a_4$$

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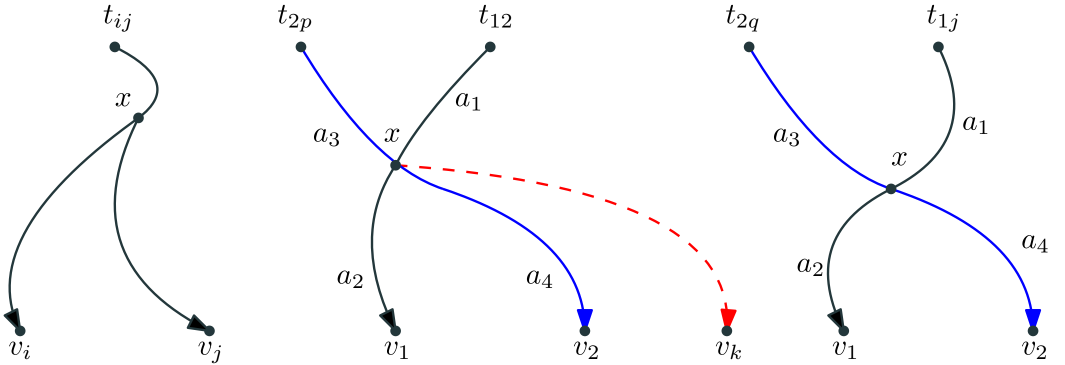
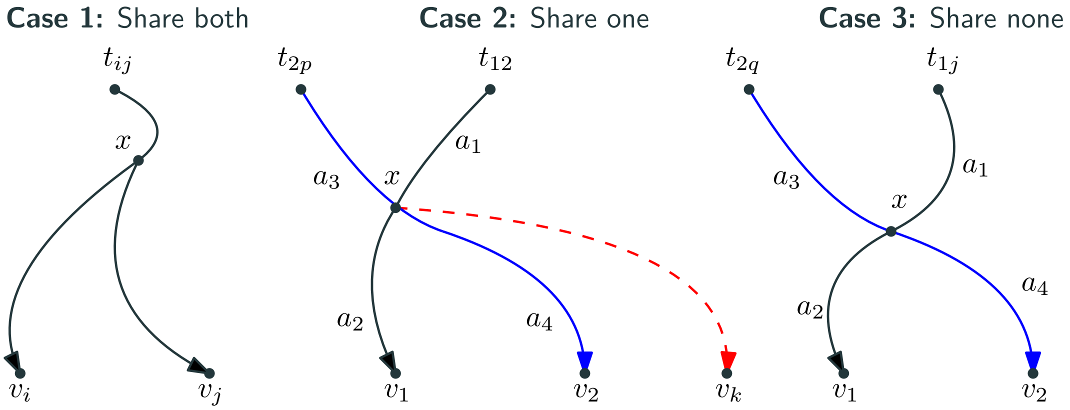


$$d(t_{1j}, v_1) + d(t_{2q}, v_2) = a_1 + a_2 + a_3 + a_4 \;\; = (a_1 + a_4) + (a_2 + a_3)$$

Non-crossing of ball paths



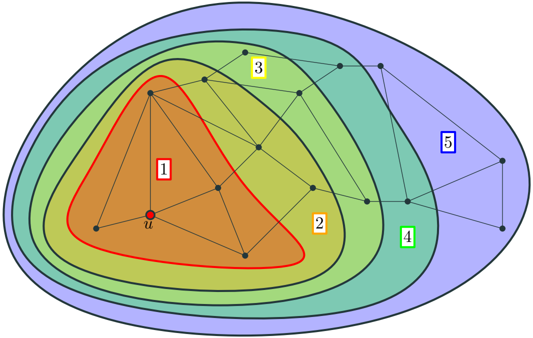
$$d(t_{1j}, v_1) + d(t_{2q}, v_2) = a_1 + a_2 + a_3 + a_4 \; = (a_1 + a_4) + (a_2 + a_3) \; > d(t_{2q}, v_2) + d(t_{1j}, v_1)$$

VC Dimension in Graphs

Definition: A ball $B(u, r)$ is the set of vertices distance $\leq r$ from $u \in V$.

Let $\mathcal{B}$ denote the set of all balls, i.e. $\mathcal{B} = \{B(u, r) \mid u \in V, r \in [n]\}$.

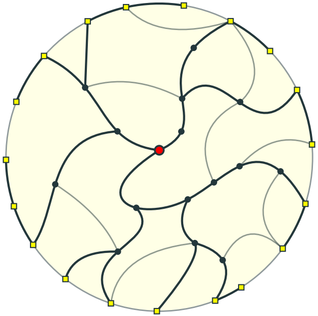
Theorem. If G is K_h-minor-free, the set $\mathcal{B}$ has VC dimension at most $h - 1$.



Other bounded VC-set systems

Another Object with Bounded VC Dimension

Planar G, outer face vertices $s_0, \ldots, s_\ell$.



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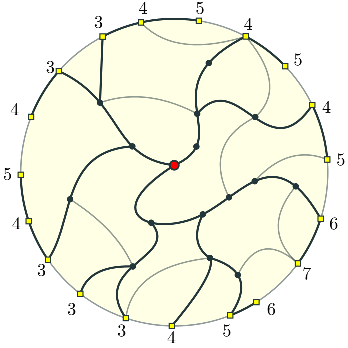

Another Object with Bounded VC Dimension

Planar G, outer face vertices $s_0, \ldots, s_\ell$.

$$X_{LP}(u) := \{(i, \delta) \mid d(u, s_i) - d(u, s_{i-1}) = \delta\}$$
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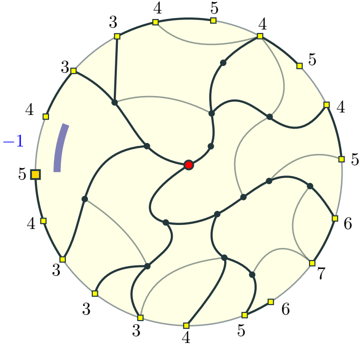

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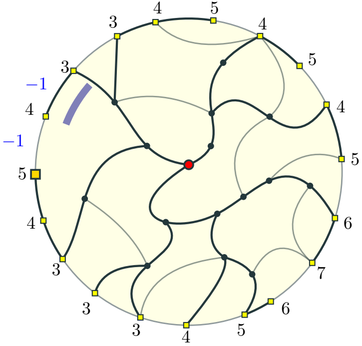

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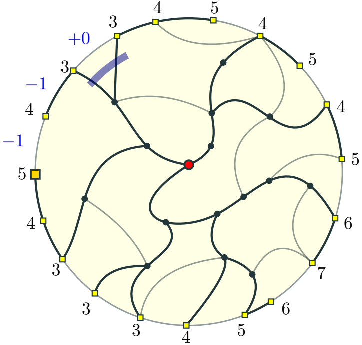

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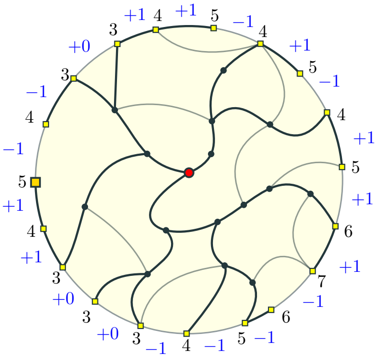
Theorem. [Li–Parter SODA '19]
If G is planar, $\mathrm{VCDim}(\mathcal{X}_{LP}) \leq 3$.

$$\begin{bmatrix} -1 \\ -1 \\ +0 \\ +1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ +1 \\ +1 \\ -1 \\ -1 \\ -1 \\ -1 \\ +0 \\ +0 \\ +1 \\ +1 \end{bmatrix}$$



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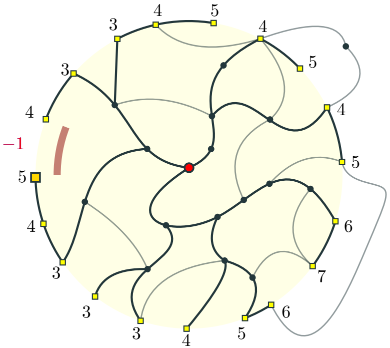

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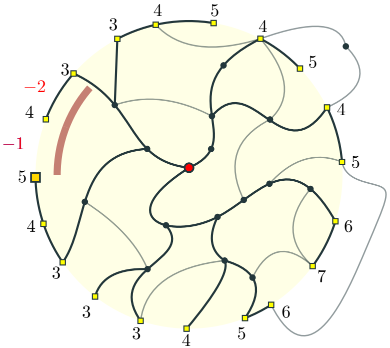

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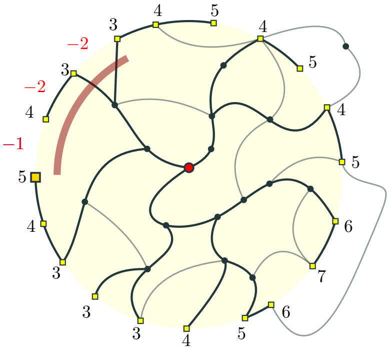

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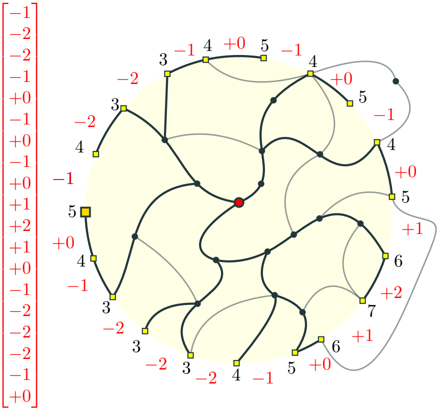

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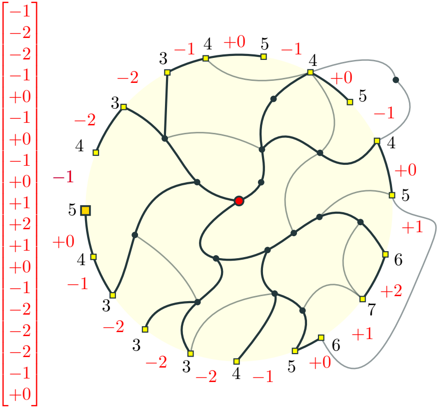

Theorem. [Le–Wulff-Nilsen SODA '24]
G is K_h-minor-free, $\mathrm{VCDim}(\overrightarrow{\mathcal{X}_\Delta}) \leq h^2$.

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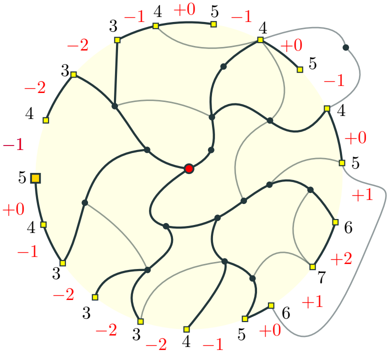

Theorem. [This work]
G is K_h-minor-free, $VCDim(\overrightarrow{\mathcal{X}_\Delta}) \leq h - 1$.

What is the set system $X_\Delta(u)$?

A Different View

$$X_\Delta(u) := \{(i,\delta) \mid \delta \in \Delta, d(u,s_i) - d(u,s_0) \le \delta\}$$

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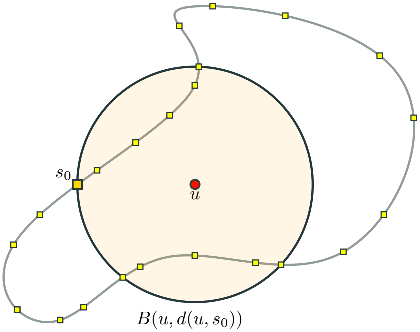
Theorem. [Le–Wulff-Nilsen SODA '24]
If G is K_h-minor-free, VCDim($\mathcal{X}_\Delta$) $\leq h-1$.

Let $\Delta = \{-1, 0, +3\}$.

A Different View



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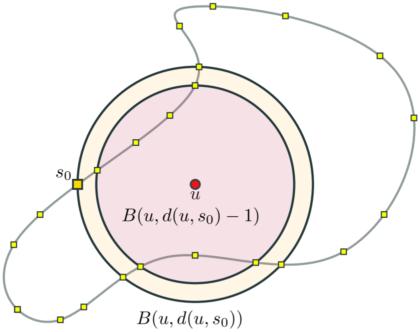

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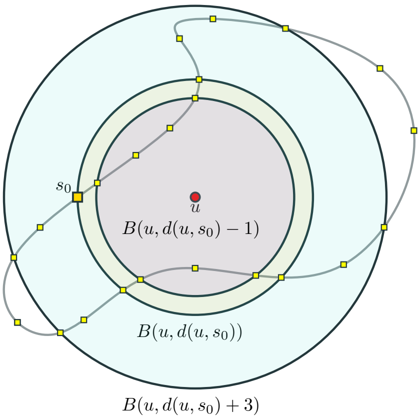

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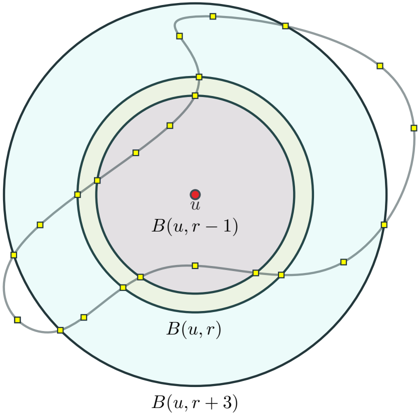

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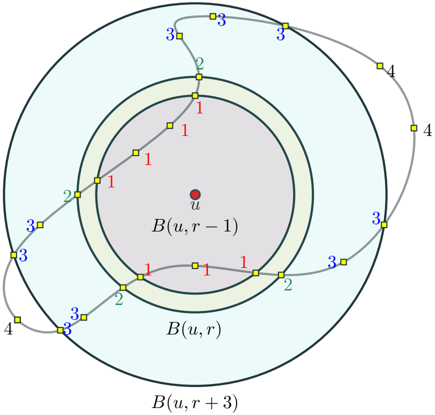
If G is K_h-minor-free, $VCDim(\mathcal{X}_\Delta) \leq h-1$.

Let $\Delta = \{-1, 0, +3\}$.

$X_\Delta(u)$ encodes information about *multiballs*!

A Different View



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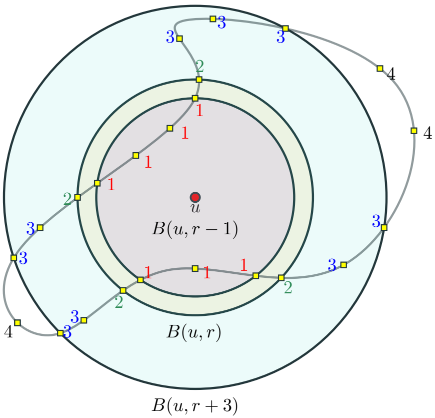

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Let MB_Δ denote multiballs of arbitrary radii.

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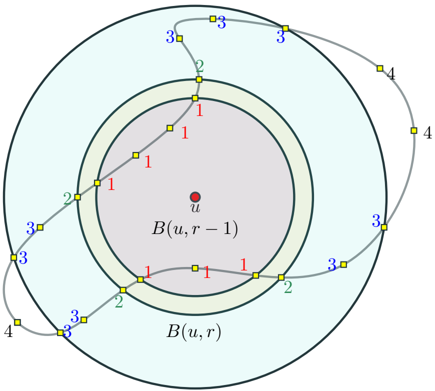

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$X_\Delta(u)$ encodes information about *multiballs*!

Let MB_Δ denote multiballs of arbitrary radii.

Theorem. [Karczmarz-**Z.** SODA '25]
If G is K_h-minor-free, PDim$(MB_\Delta) \leq h-1$.

Why is this useful?

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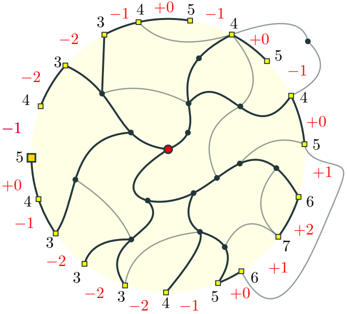
Theorem. [Le–Wulff-Nilsen SODA '24] *If G is K_h-minor-free, VCDim$(\mathcal{X}_\Delta) \leq h - 1$.*

$$\begin{bmatrix} -1 \\ -2 \\ -2 \\ -1 \\ +0 \\ -1 \\ +0 \\ -1 \\ +0 \\ +1 \\ +2 \\ +1 \\ +0 \\ -1 \\ -2 \\ -2 \\ -2 \\ -1 \\ +0 \end{bmatrix}$$



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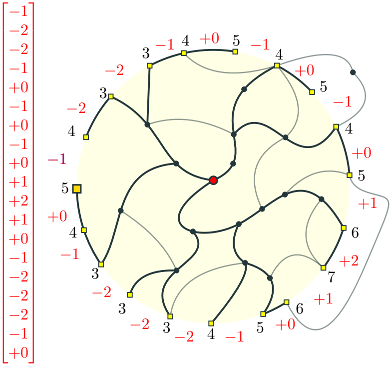
Theorem. [Le–Wulff-Nilsen SODA '24] *If G is K_h-minor-free, $VCDim(\mathcal{X}_\Delta) \leq h - 1$.*

Theorem. [Sauer's Lemma] *If $VCDim(X, \mathcal{R}) = d$, then $|\mathcal{R}| \leq O(|X|^d)$*



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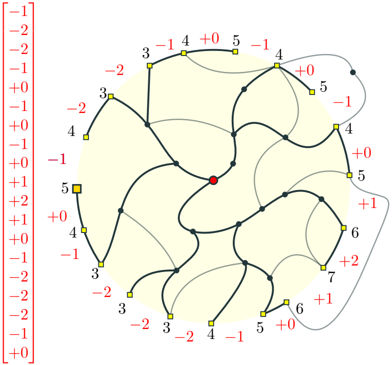
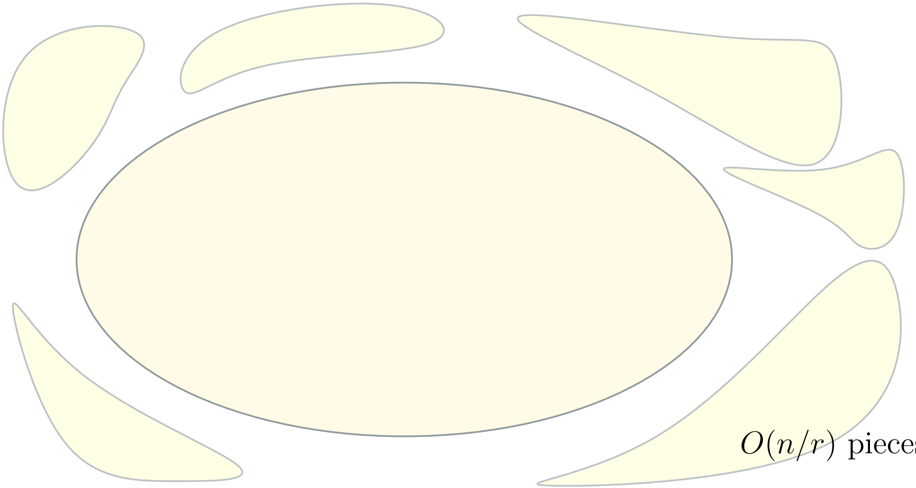
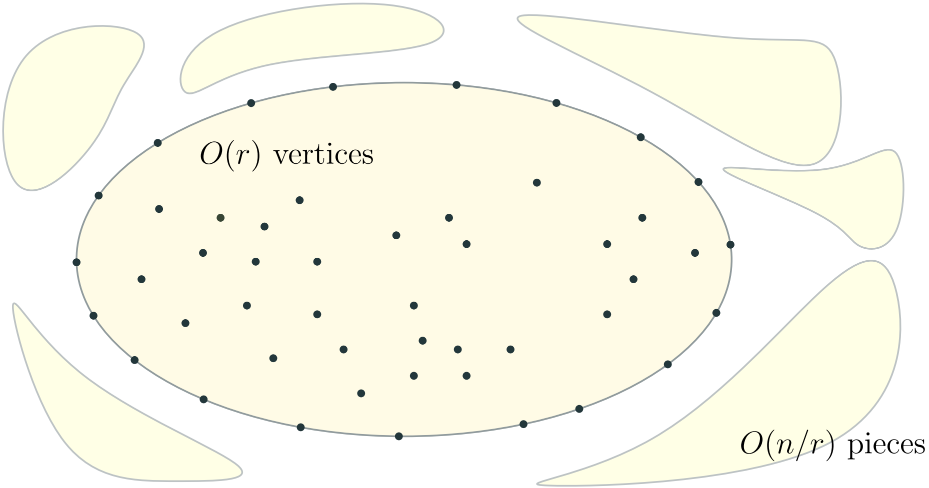
Theorem. [Sauer's Lemma] *If $VCDim(X, \mathcal{R}) = d$, then $|\mathcal{R}| \leq O(|X|^d)$*

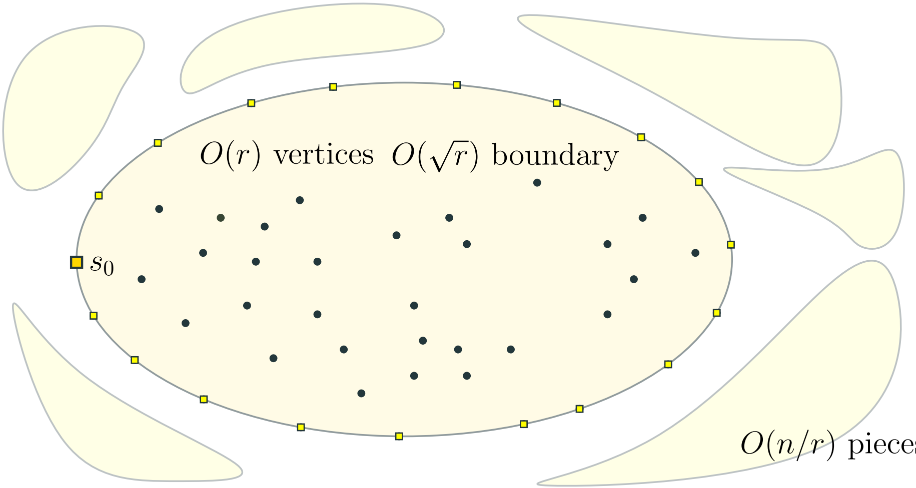
Implies number of vectors is at most $(2 \cdot \text{Diameter}(G) \cdot |\text{Outer face}|)^d$.



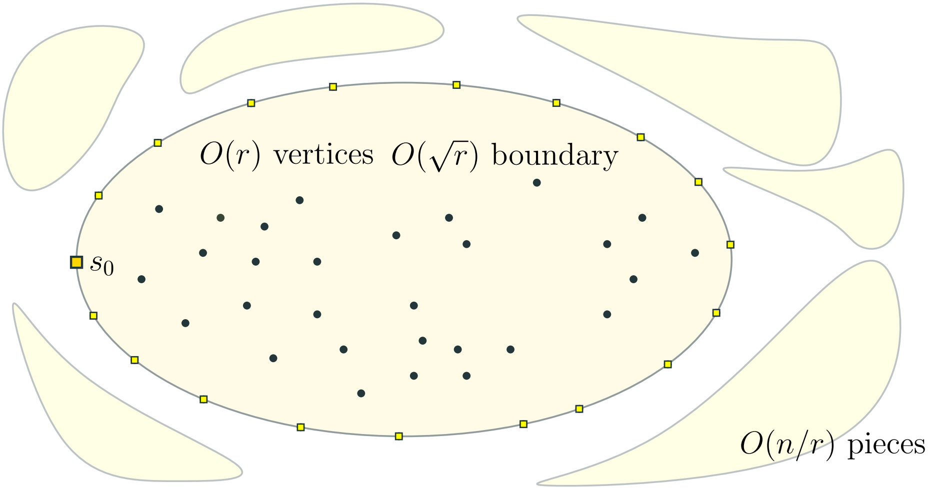
Warm up: A simple distance oracle


$O(n/r)$ pieces

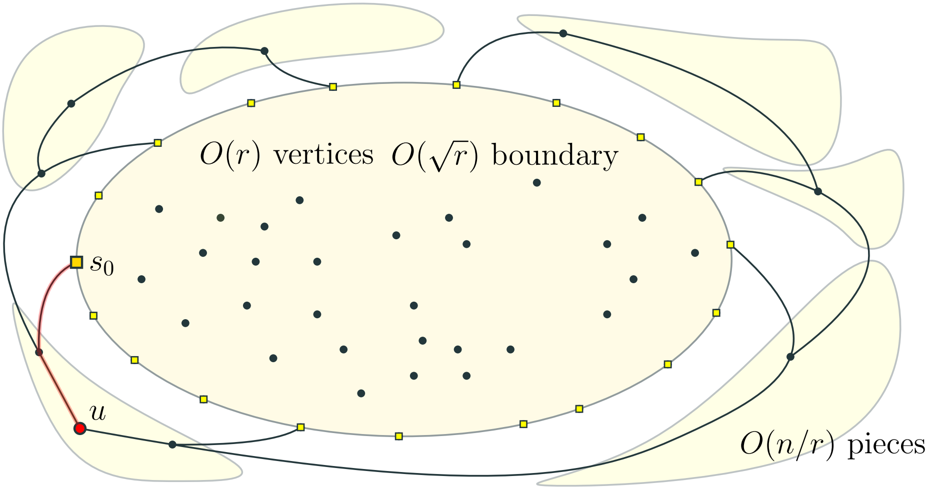

$O(r)$ vertices
$O(n/r)$ pieces


$O(r)$ vertices $O(\sqrt{r})$ boundary
s_0
$O(n/r)$ pieces

BFS: $O(n^2/\sqrt{r})$ distances



BFS: $O(n^2/\sqrt{r})$ distances



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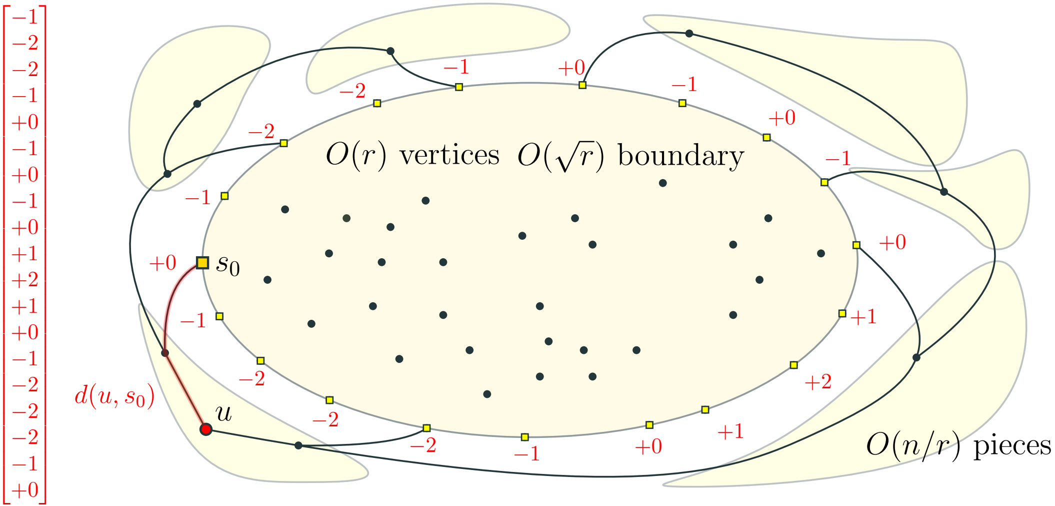
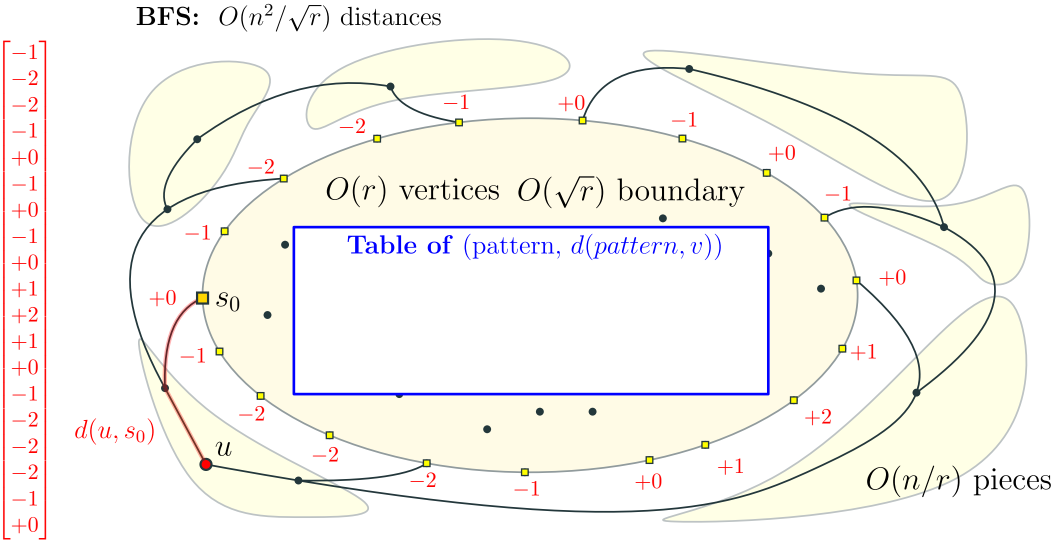
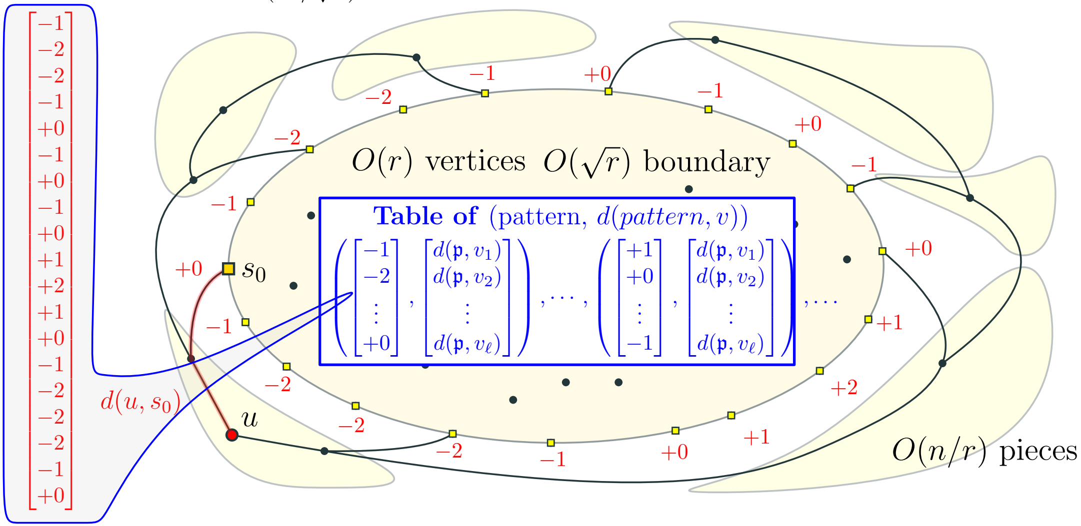


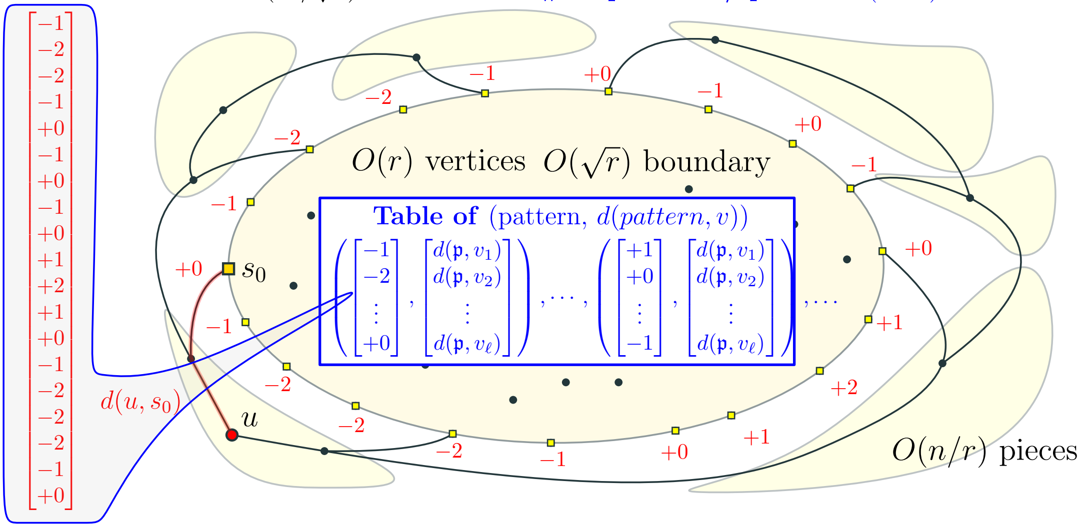


BFS: $O(n^2/\sqrt{r})$ distances
$O(r)$ vertices $O(\sqrt{r})$ boundary
Table of (pattern, $d(pattern, v)$)
s_0
u
$d(u, s_0)$
$O(n/r)$ pieces
$[-1, -2, -2, -1, +0, -1, +0, -1, +0, +1, +2, +1, +0, -1, -2, -2, -2, -1, +0]$

BFS: $O(n^2/\sqrt{r})$ distances



BFS: $O(n^2/\sqrt{r})$ distances **# of patterns / piece:** $O(r^{O(1)})$



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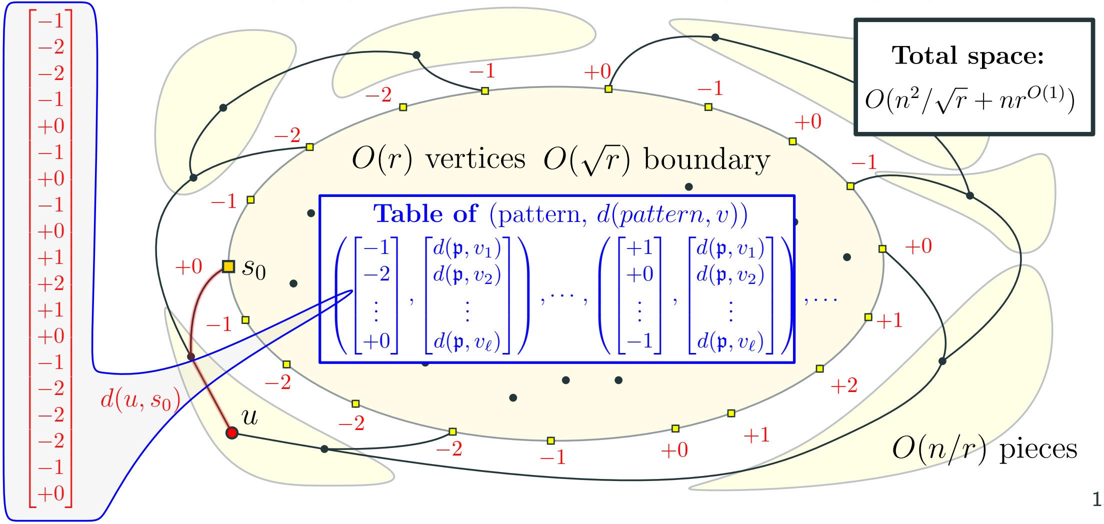

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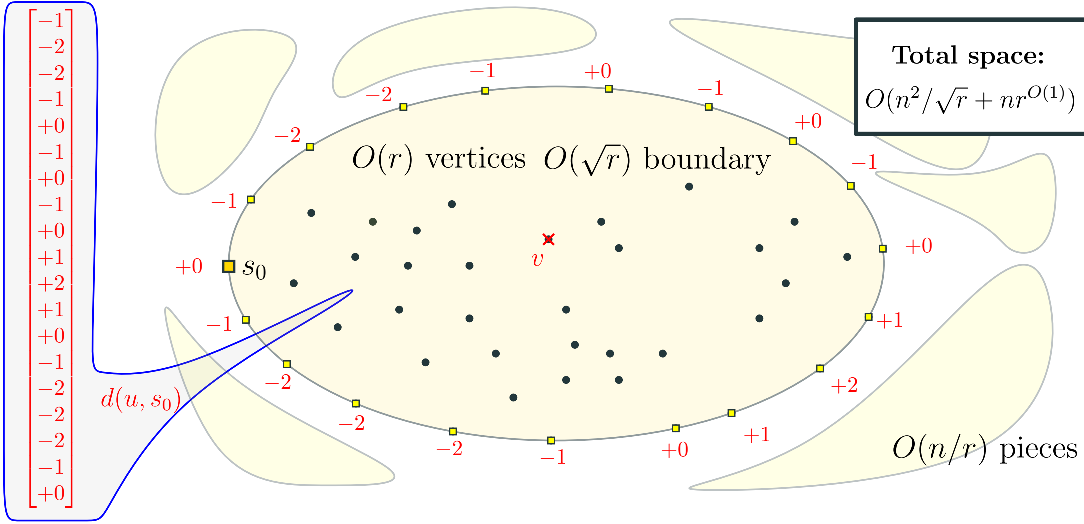

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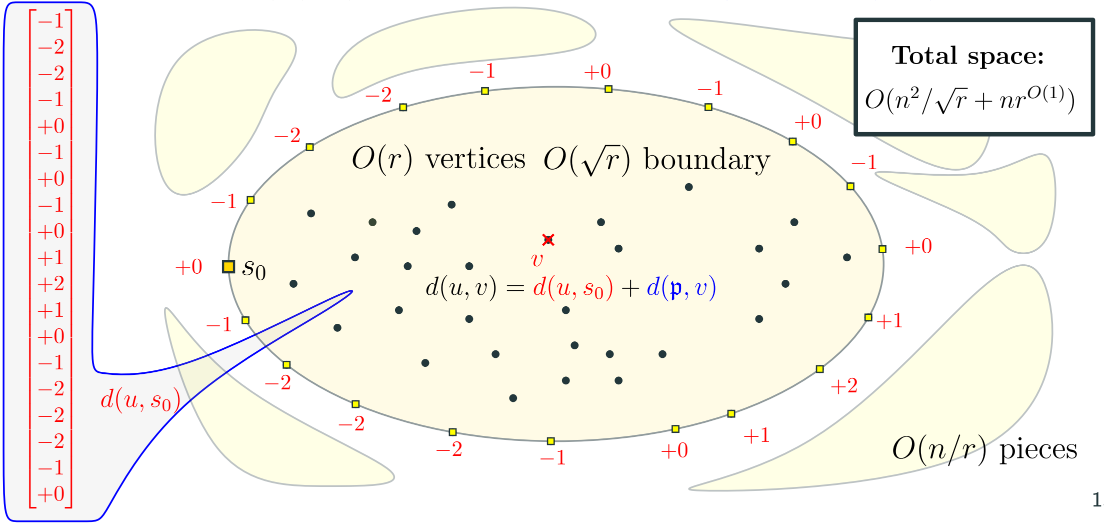

An algorithmic template

Basic Algorithmic Template

1. r-division of G into $O(n/r)$ pieces of size $O(r)$ with $O(n/\sqrt{r})$ boundary.
2. Store patterns to boundary of each piece by BFS using $O(n^2/\sqrt{r})$ space.
3. Store $r^{O(1)}$ relevant patterns per piece and other precomputation (e.g. distances of patterns to vertices). Overall space $O(n/r) \cdot O(r^{O(1)}) = O(nr^{O(1)})$.

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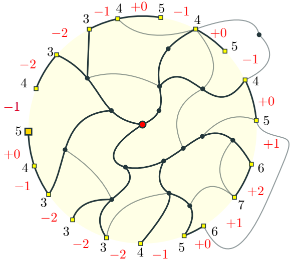
Real-weighted distance oracle

Technical Challenge

$$X_\Delta(u) := \{(i,\delta) \mid \delta \in \Delta, d(u,s_i) \leq d(u,s_0) + \delta\} \qquad \mathcal{X}_\Delta := \{X_\Delta(u) \mid u \in V\}$$

Theorem. [Le–Wulff-Nilsen SODA '24] *If G is K_h-minor-free, $VCDim(\mathcal{X}_\Delta) \leq h-1$.*

$$\begin{bmatrix} -1 \\ -2 \\ -2 \\ -1 \\ +0 \\ -1 \\ +0 \\ -1 \\ +0 \\ +1 \\ +2 \\ +1 \\ +0 \\ -1 \\ -2 \\ -2 \\ -2 \\ -1 \\ +0 \end{bmatrix}$$



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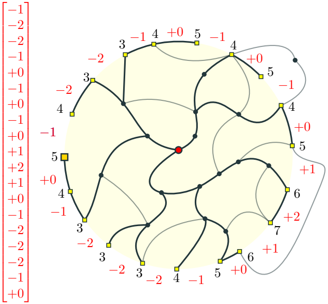

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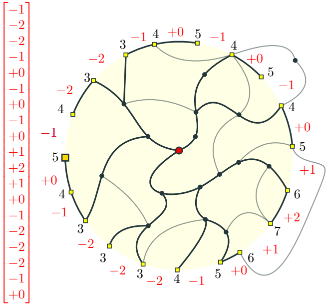
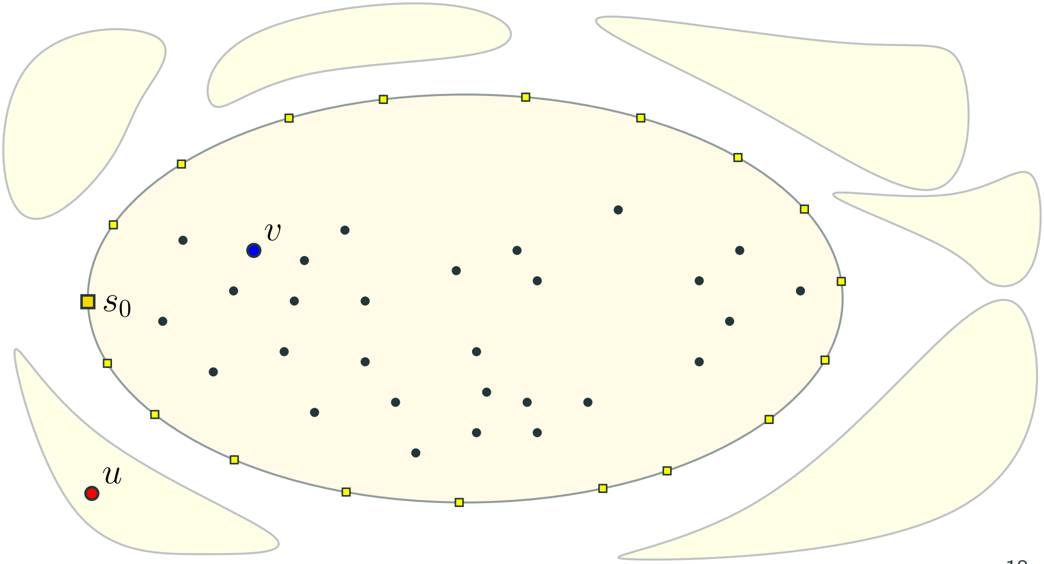
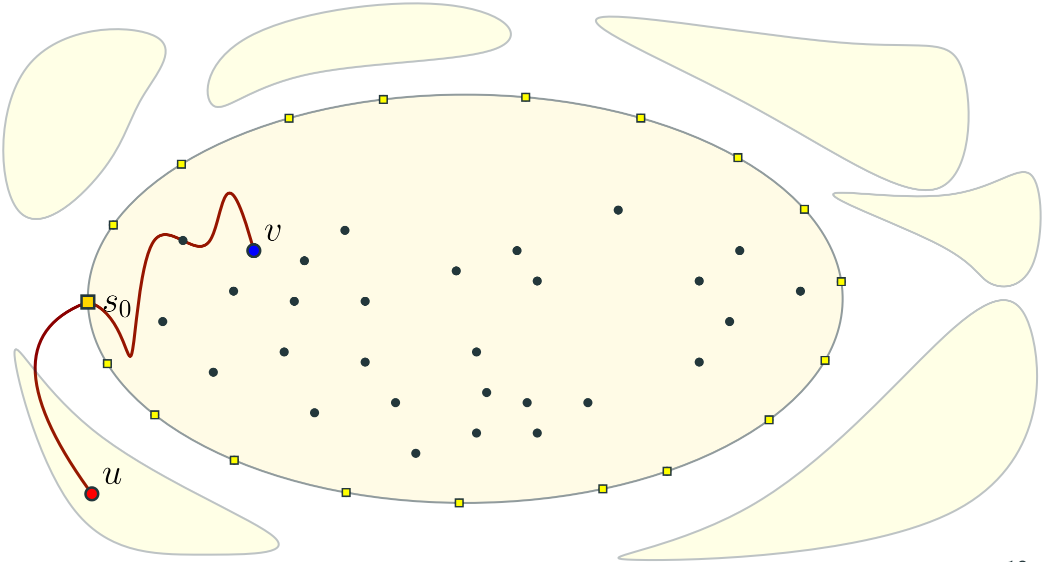
Implies number of vectors is at most $(|\Delta| \cdot |\text{Outer face}|)^d$.





v
s_0
u


v
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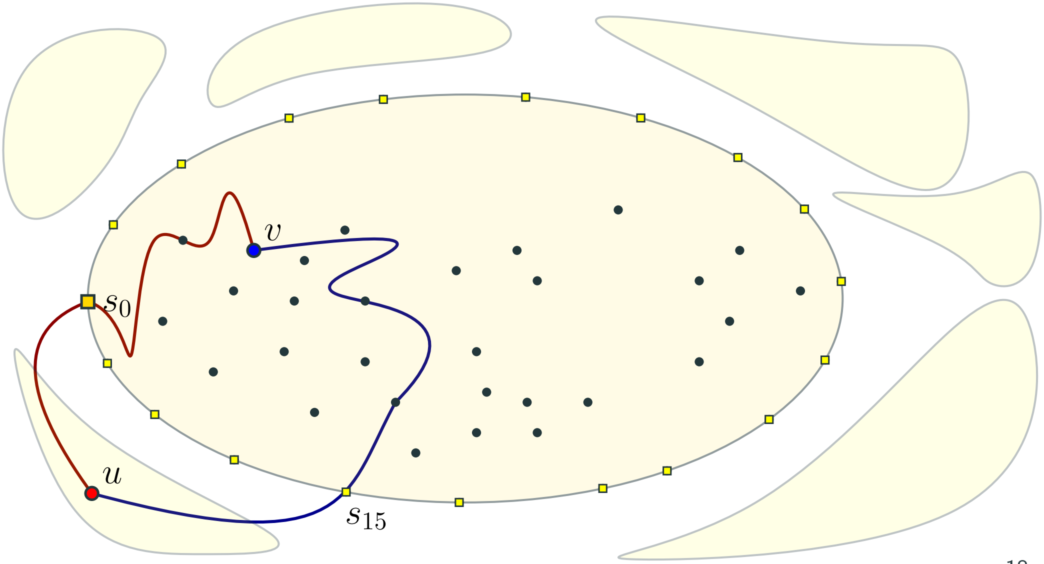
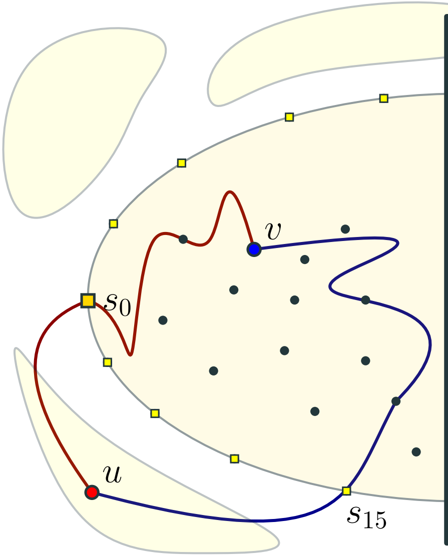
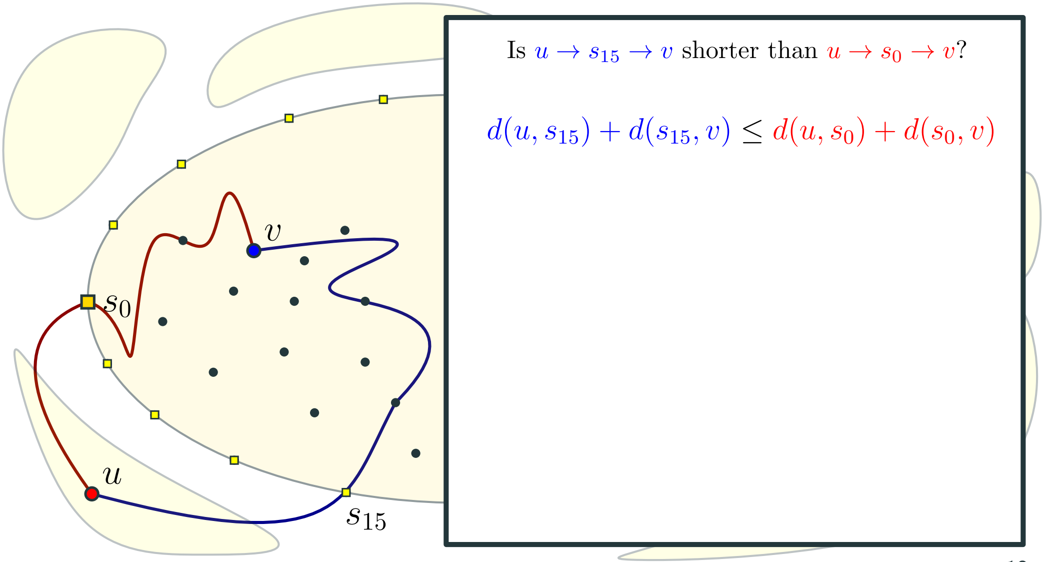

v
s_0
u
s_{15}



Is $u \to s_{15} \to v$ shorter than $u \to s_0 \to v$?

$$d(u, s_{15}) + d(s_{15}, v) \leq d(u, s_0) + d(s_0, v)$$



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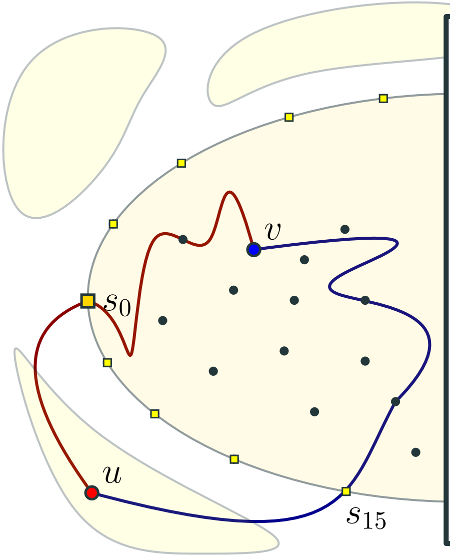

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if and only if

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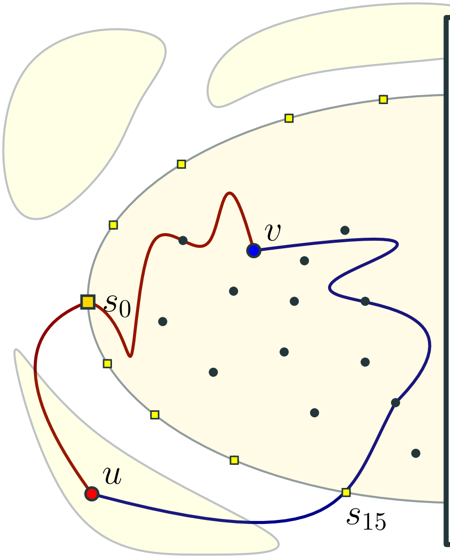

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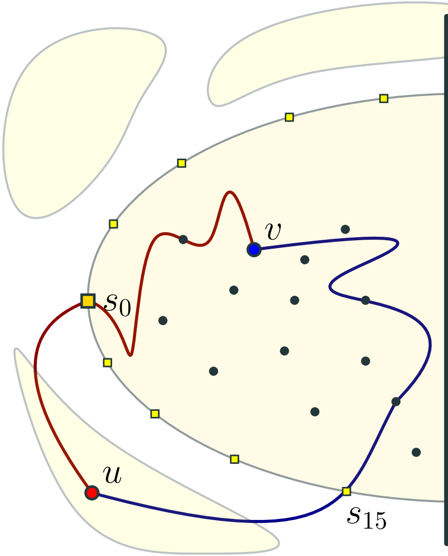
"Fredman's trick" [Fredman '76]



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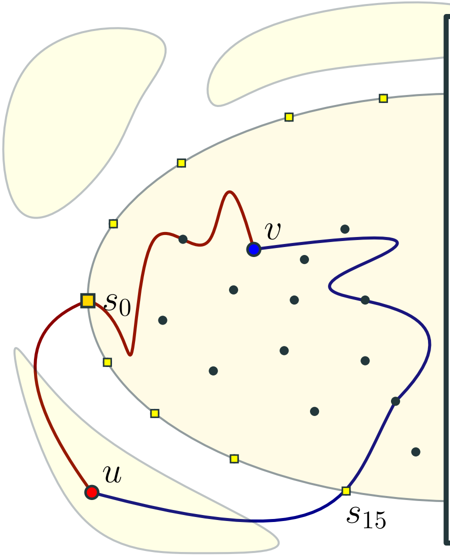

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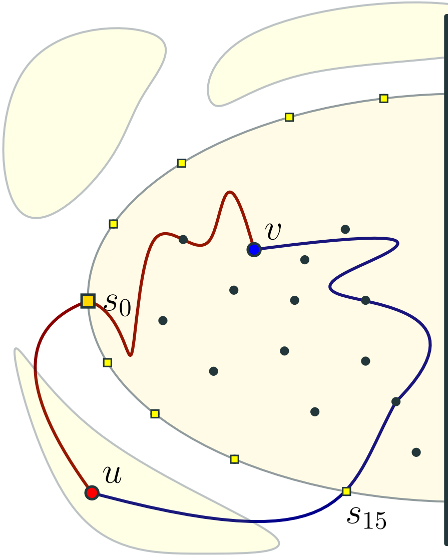

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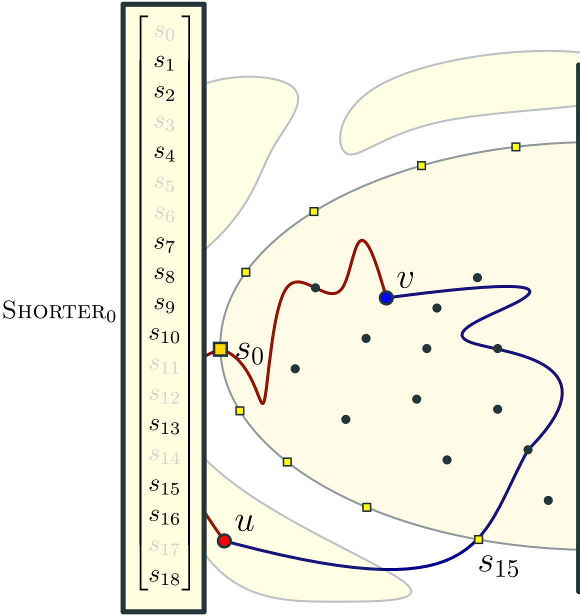

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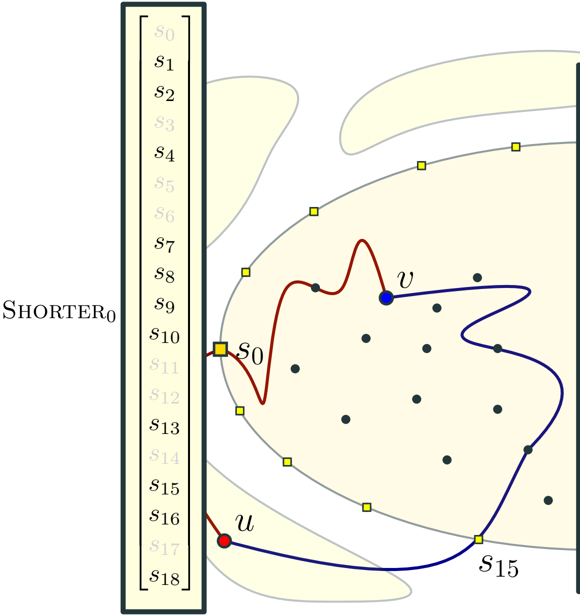

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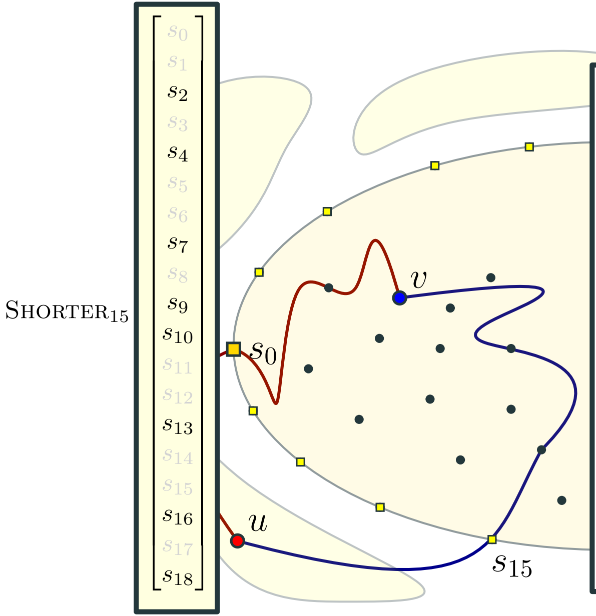

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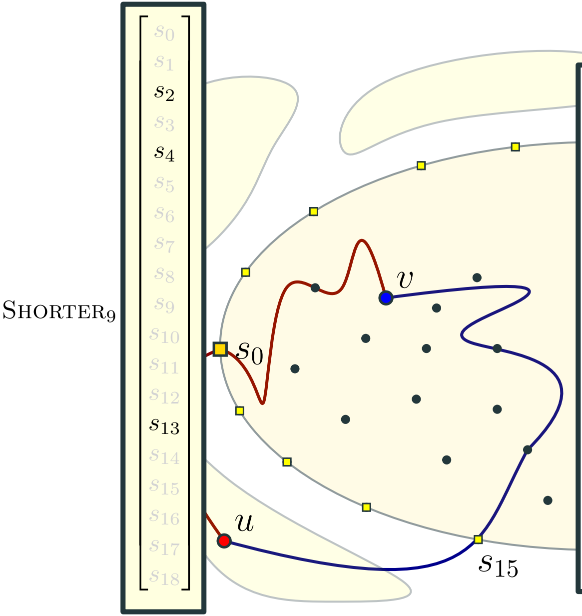

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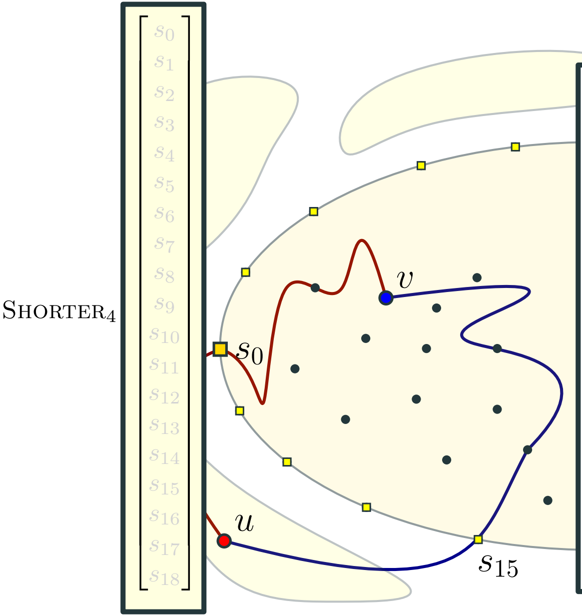

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Open Question 2: Using other set systems of bounded VC dimension?

(e.g. VC dimension in general graphs Kleinberg [FOCS '00])

What is the set system $X_\Delta(u)$?

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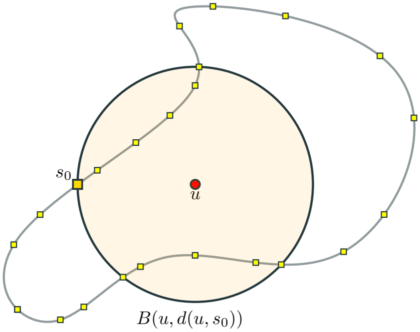

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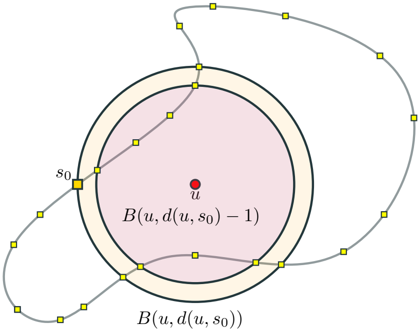

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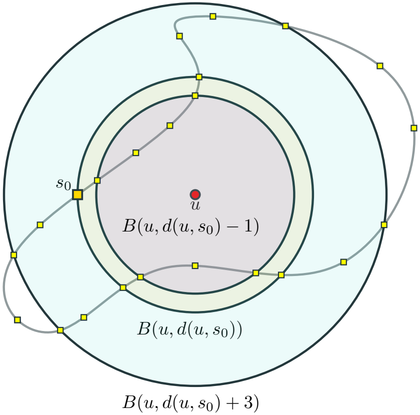

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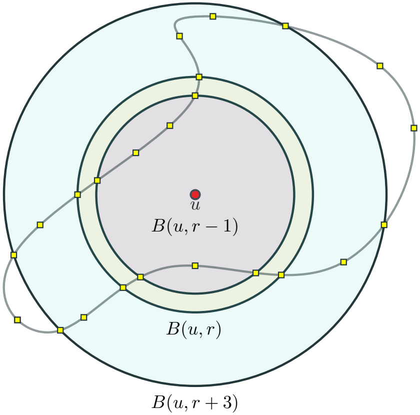

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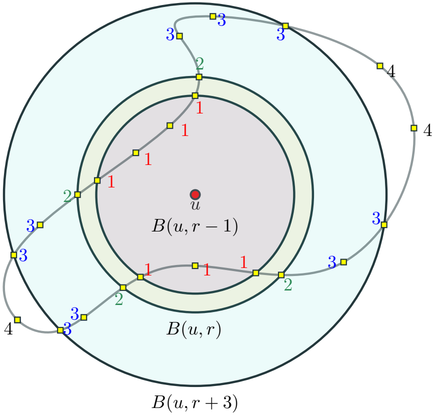

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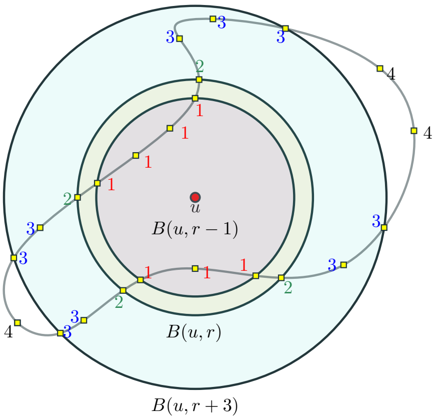

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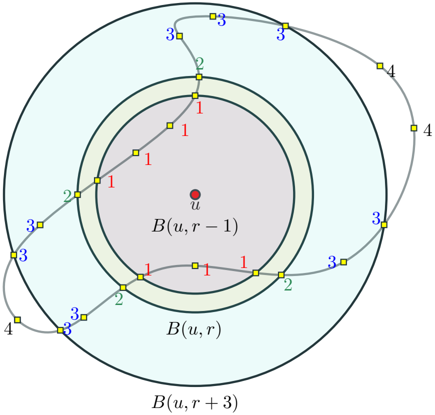

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Theorem. [Karczmarz-**Z.** SODA '25]
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I understand what MB_Δ is!

Summary of results and open problems

- Exact distance oracles in real-weighted K_h-minor-free digraphs with $O(\log n)$ query time and $O(n^{2-1/(4h-1)})$ space.
- Improved understanding of the $\mathcal{X}_\Delta$, reducing VC dimension of directed K_h-minor-free graphs to $h-1$.
- Decremental reachability oracles with $\widetilde{O}(n^{2-1/h})$ total update time.

  New even for planar digraphs!.
- In unweighted digraphs: subquadratic constructions of exact distance oracles, vertex eccentricities, and Weiner index, using string algorithms.

Open Question: What else can we do in graphs with bounded distance VC dimension? Subquadratic time real-weighted diameter?

Open Question 2: Using other set systems of bounded VC dimension?

(e.g. VC dimension in general graphs Kleinberg [FOCS '00])