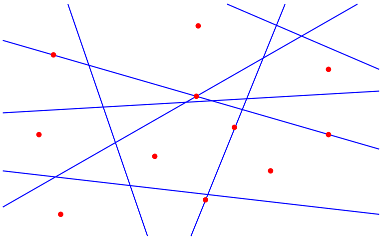
Hopcroft's Problem

2D Fractional Cascading and Decision Trees



Timothy M. Chan and **Da Wei Zheng**

January 9, 2022

University of Illinois Urbana-Champaign

Outline

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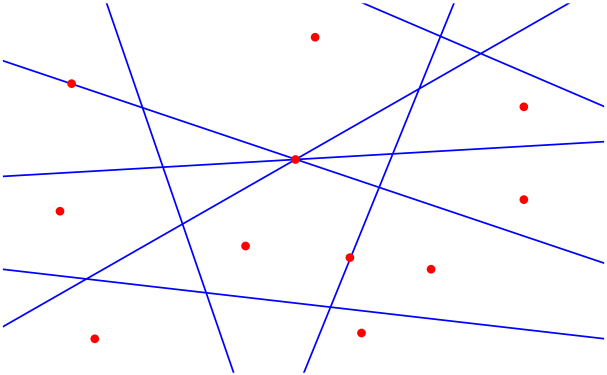

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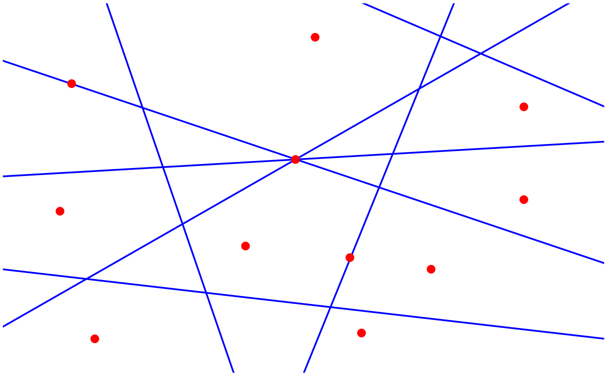

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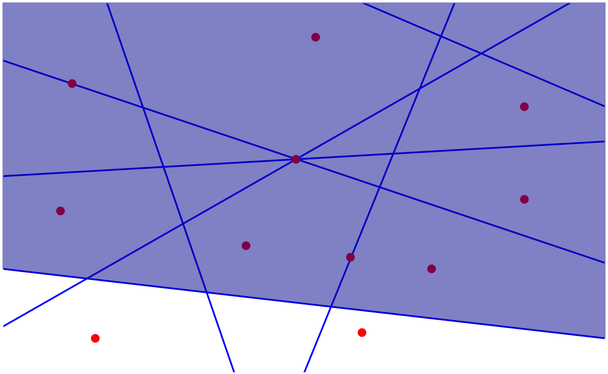

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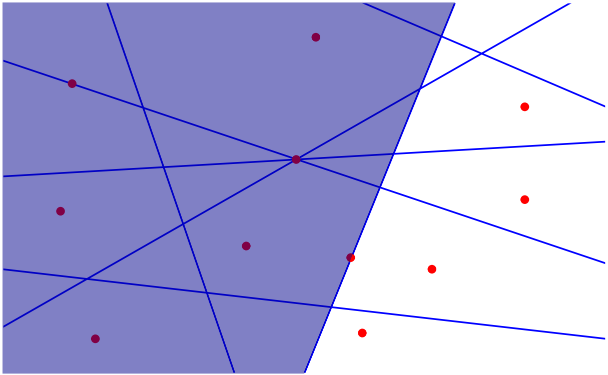

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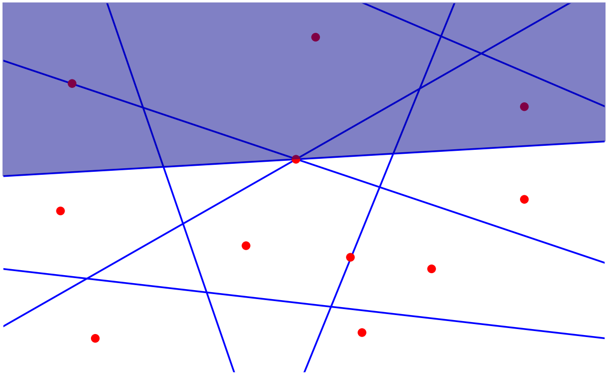

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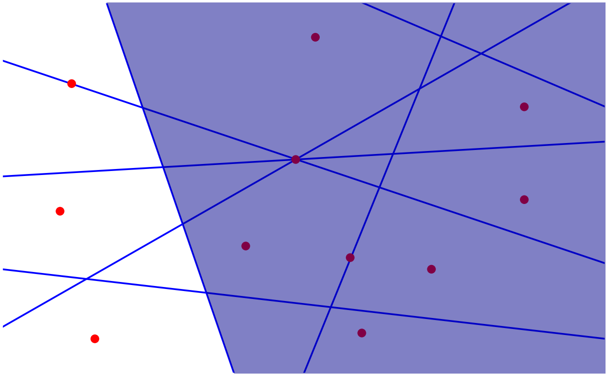

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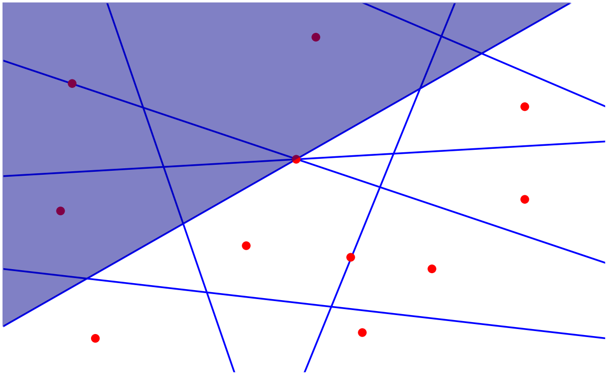

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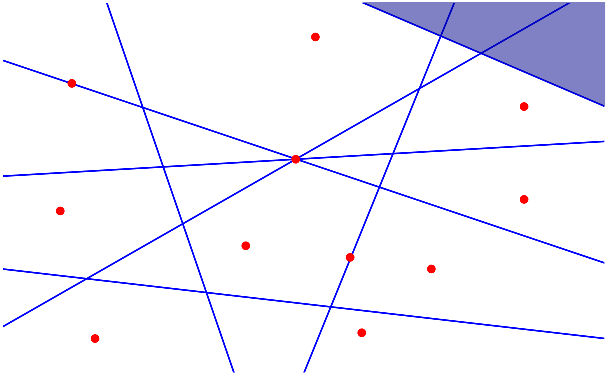

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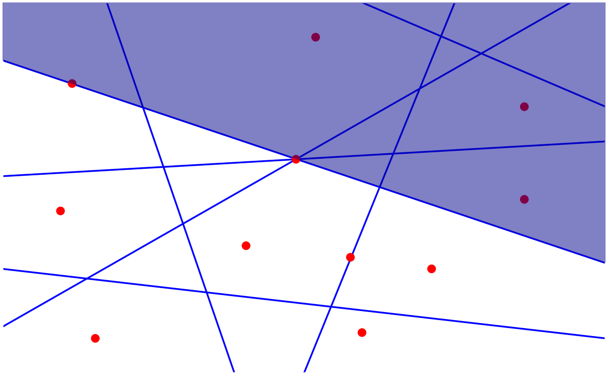

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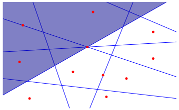

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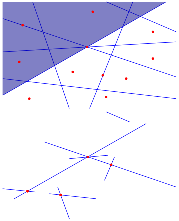

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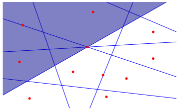

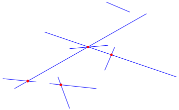

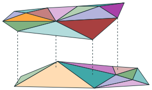

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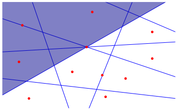
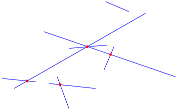
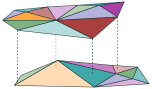
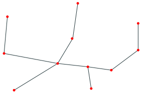

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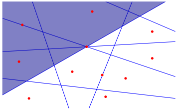

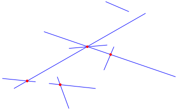

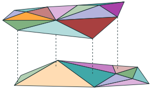

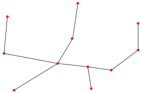

... and many other problems in computational geometry!

Outline

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$O(n^{4/3})$ algorithm for Hopcroft's problem **NEW!**

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- Offline simplex range query [Matoušek, '93] $O(n^{2d/(d+1)}2^{O(\log^* n)})$
- 2D line segment intersection counting [Chazelle, '83] $O(n^{4/3}\log^{1/3} n)$
- 2D line segment connected components [Lopez, Thurimella, '85] $O(n^{4/3}\log^{3} n)$
- 3D line towering problem. [Chazelle, Edelsbrunner, Guibas, Sharir, '94] $O(n^{4/3+\varepsilon})$
- 3D vertical distance between polyhedral terrains [↑↑↑↑↑, '94] $O(n^{4/3+\varepsilon})$
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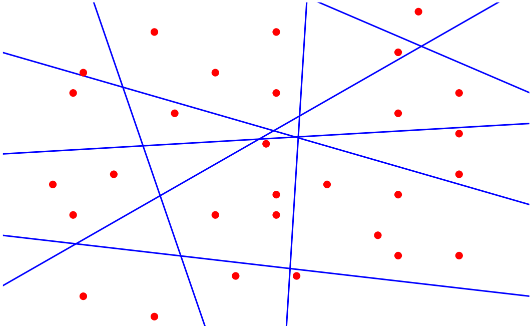

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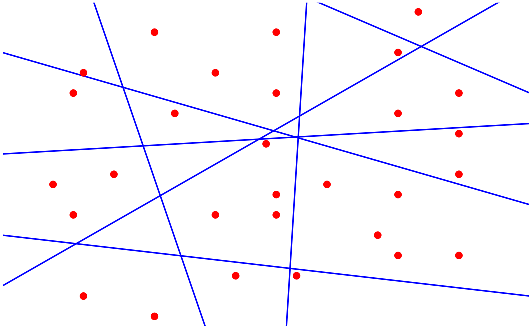

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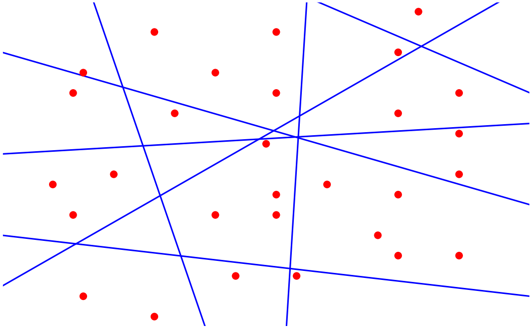

Point Location Data structure - There exists an $O(n^2)$ data structure that allows for point location queries in $O(\log n)$ time , so $T(m, n) = O(n^2 + m \log n)$.

More lines than points

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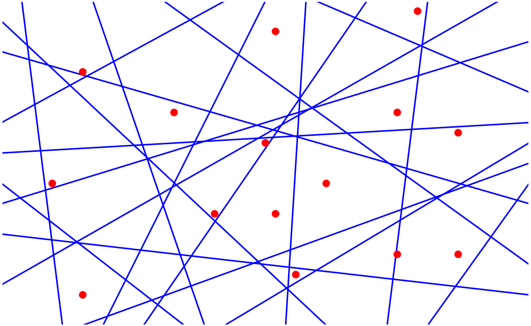

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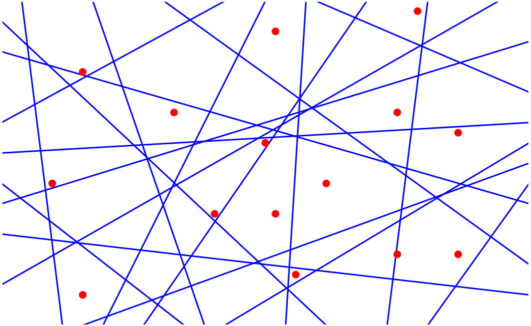
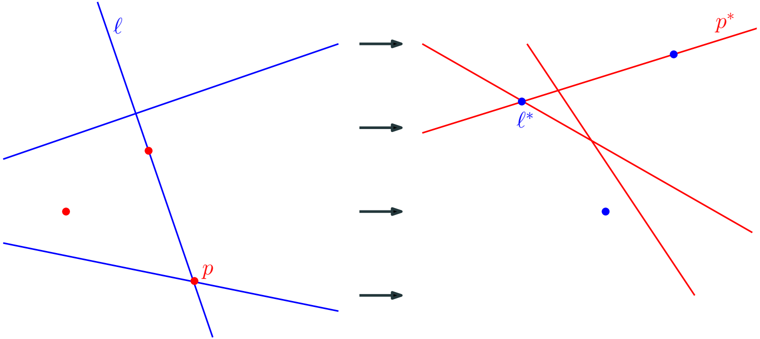


It would be nice if we can exchange our lines with our points.

Point-Line Duality

Point-Line Duality - There exists a transform that takes points to lines and lines to points that preserves incidences and above-below relationships.



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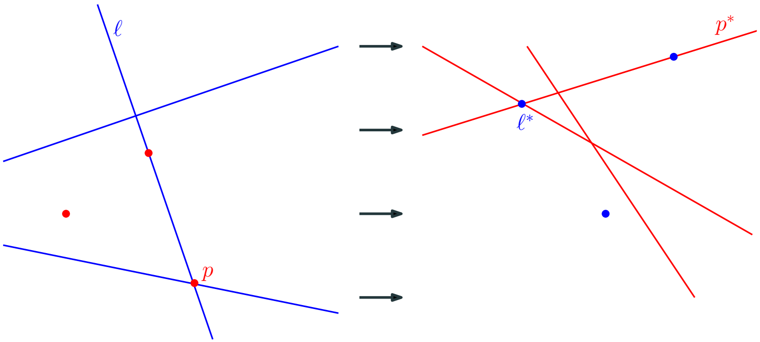

$$T(m, n) = T(n, m)$$

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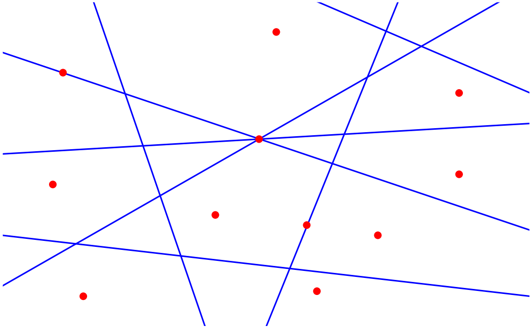
What if we have roughly equal number of lines and points, say $\sqrt{m} < n < m^2$?



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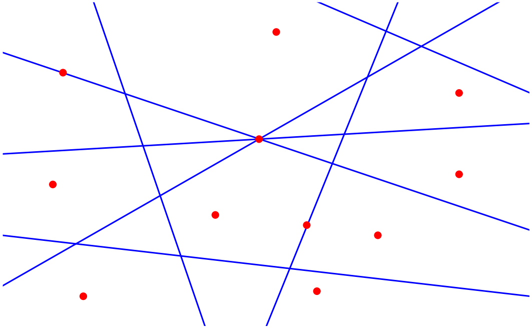
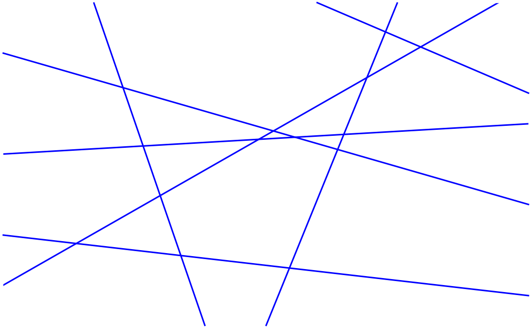


Divide and conquer?

2D Divide and Conquer

Cuttings - Given n lines and $r < n$, there exists a decomposition of $\mathbb{R}^2$ into $O(r^2)$ cells each with at most $\frac{n}{r}$ lines crossing each cell



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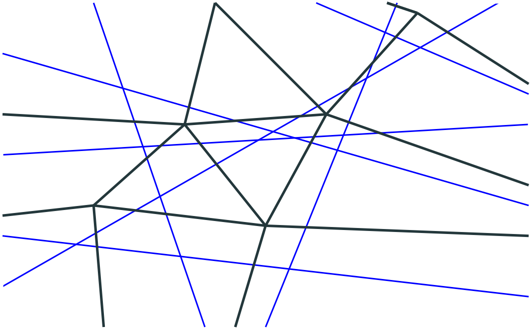

We can find these $1/r$-cuttings in time $O(nr)$.

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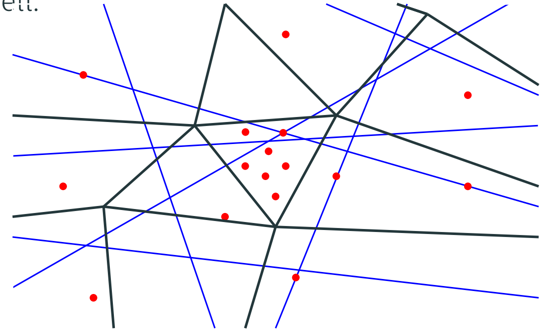

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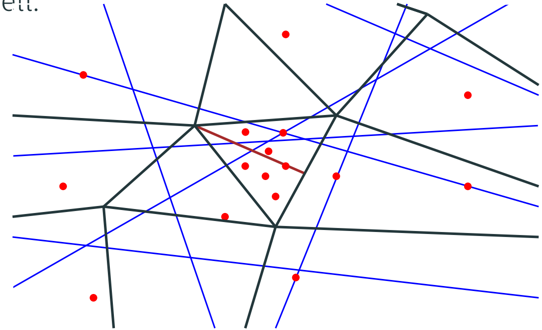

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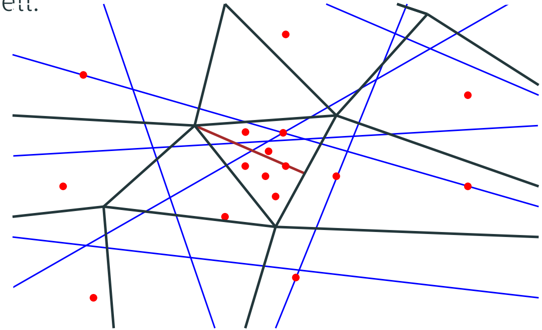

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Now we can decompose the problem: $T(m, n) = O(r^2)T\left(\frac{m}{r^2}, \frac{n}{r}\right) + O(nr + m \log r)$.

Applying cuttings to Hopcroft's problem [Chazelle, 1993]

Let $T(m, n)$ be the time to solve Hopcroft's problem with m points and n lines.

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Slightly better with $r = n^{1/3} \log^{1/3} n$ to get $O(n^{4/3} \log^{1/3} n)$ [Chazelle, 1993]

A (rederivation of) Matoušek's $n^{4/3}2^{O(\log^* n)}$ time algorithm

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Solving this will give:

$$T(n,n) = O(n^{4/3}2^{O(\log^* n)})$$

Getting rid of extra factors?

Chazelle's approach:

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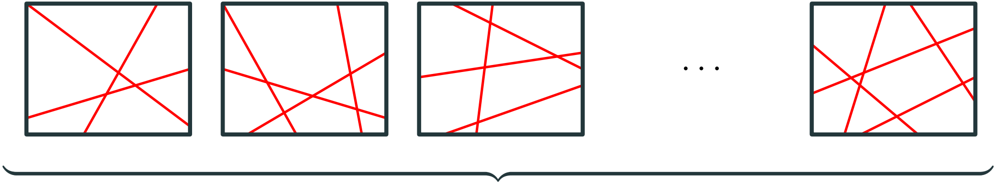

$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines

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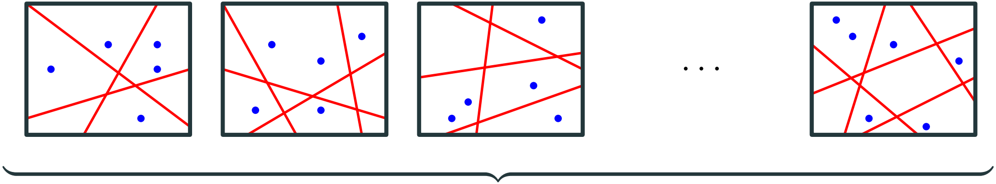

$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines and $O(n^{2/3})$ points.

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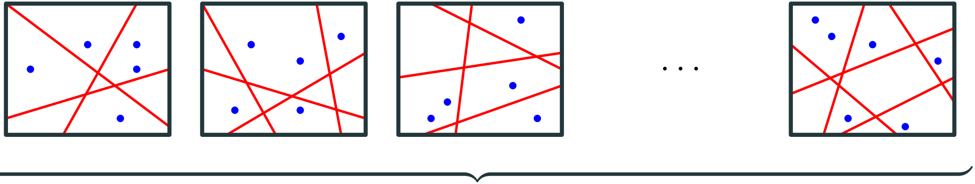

$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines and $O(n^{2/3})$ points.

$O(n^{4/3})$ point locations queries total! $\Omega(\log n)$ lower bound for doing a single point query. Can we do this faster than $O(n^{4/3}\log n)$?

Answer

Yes, we can!

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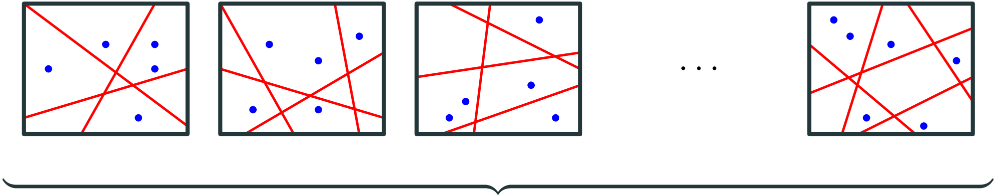

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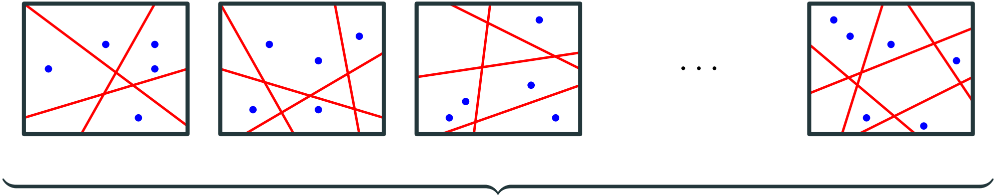


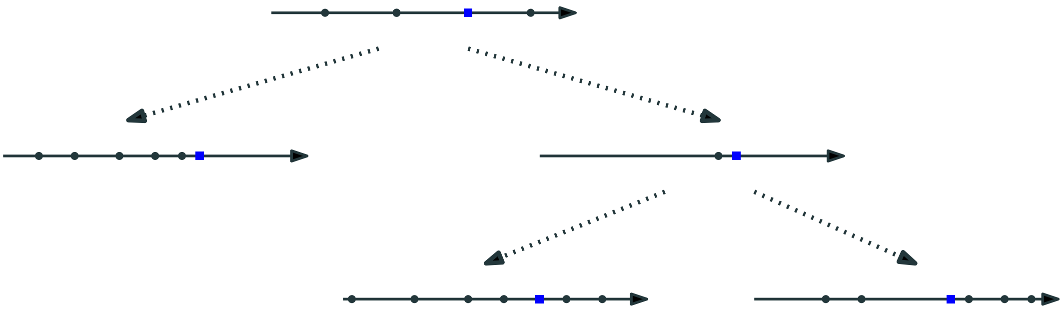
$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines and $O(n^{2/3})$ points.

Point location of n (dual) points in (average of) $O(n^{1/3})$ (dual) arrangements.

Outline

Fractional cascading in 1d lists [Chazelle, Guibas, 1986]

Suppose we're given a constant degree tree T of lists of size z and a query point p.



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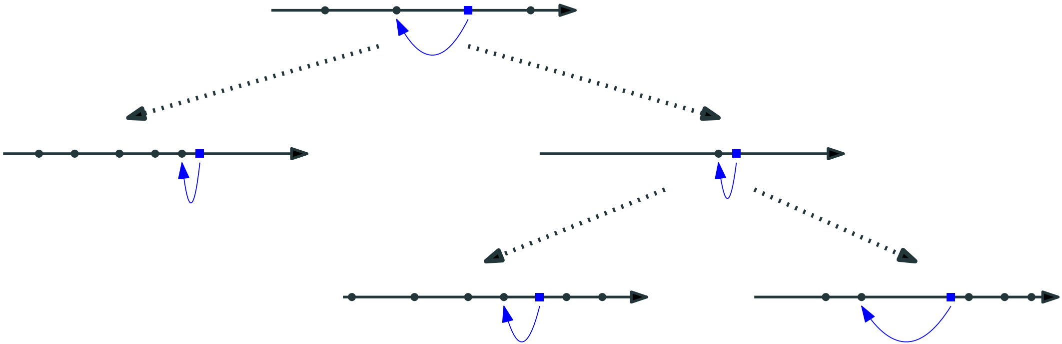

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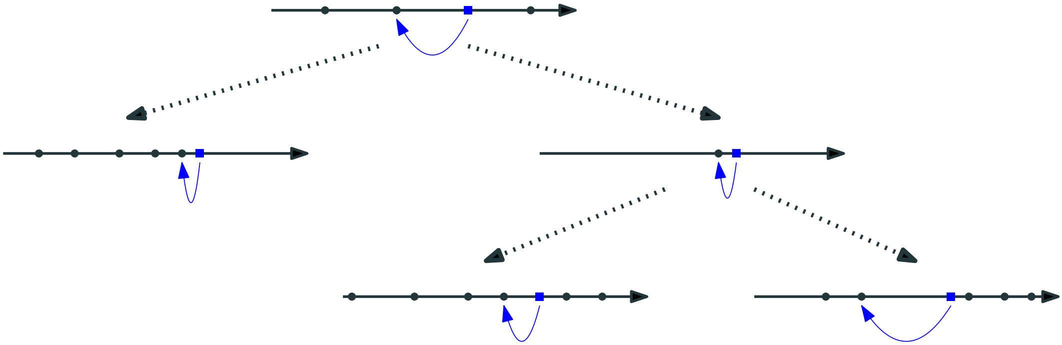
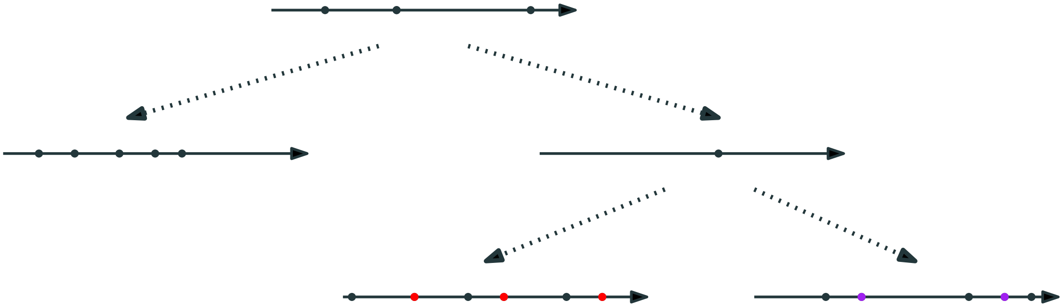
Fractional cascading finds all predecessors of p in time $O(|T| + \log z)$, this is amortized $O(1)$ per list.



Fractional cascading in 1d lists

Idea: Pass fraction $1/c$ of elements from child lists to parent lists.



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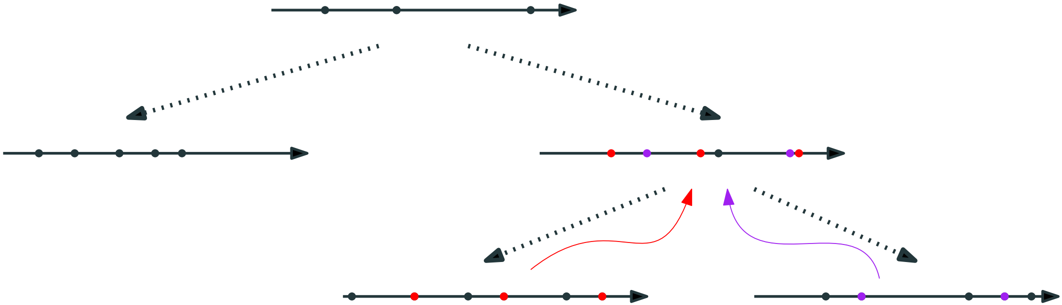

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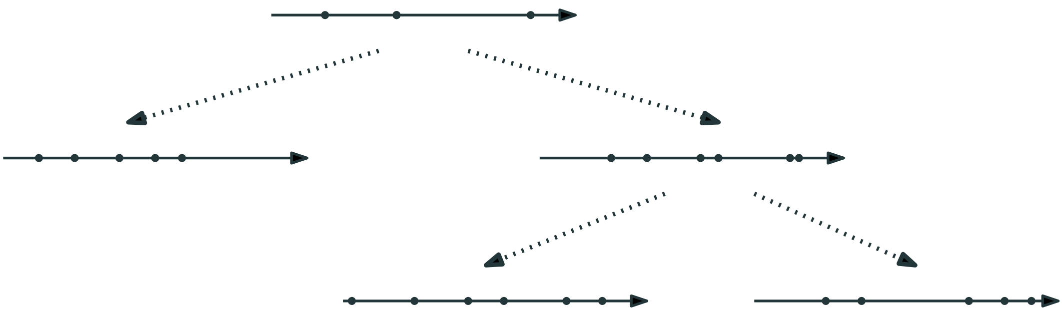

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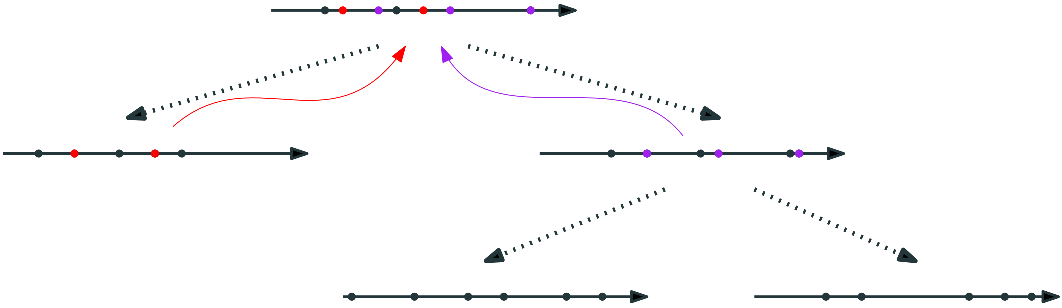

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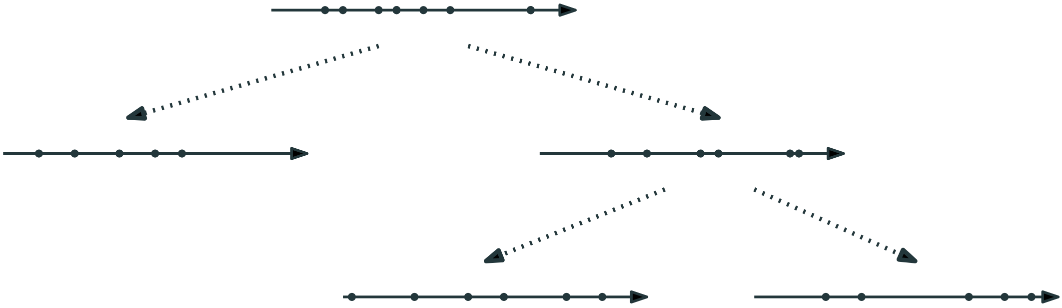

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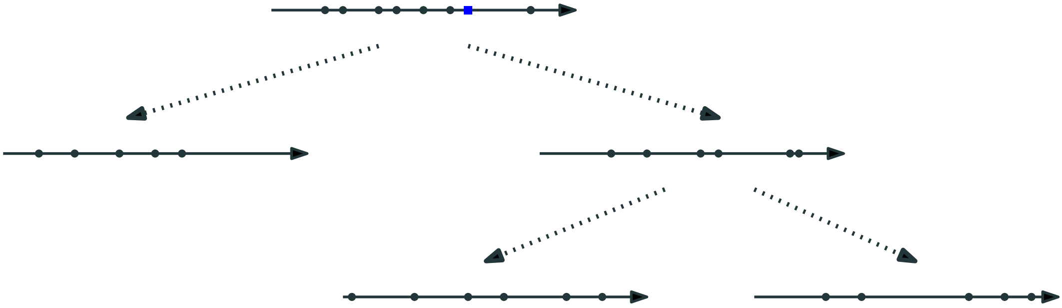

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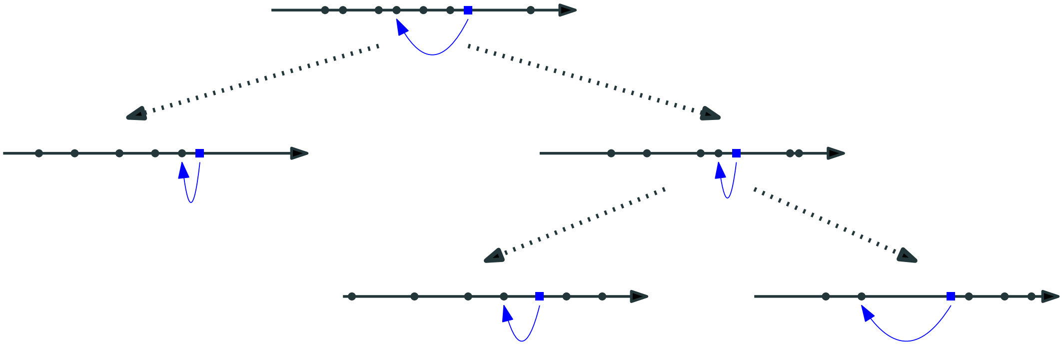

Fractional cascading in 2D?

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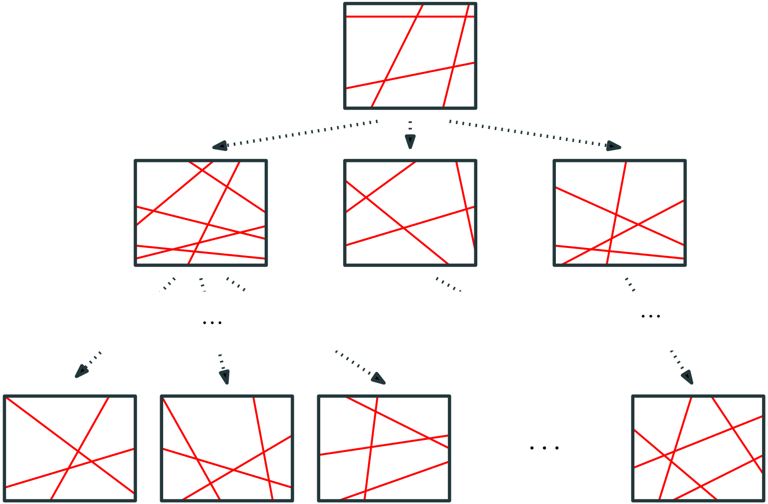
However, not general planar subdivisions, these are arrangements of lines!

Fractional cascading of line arrangements



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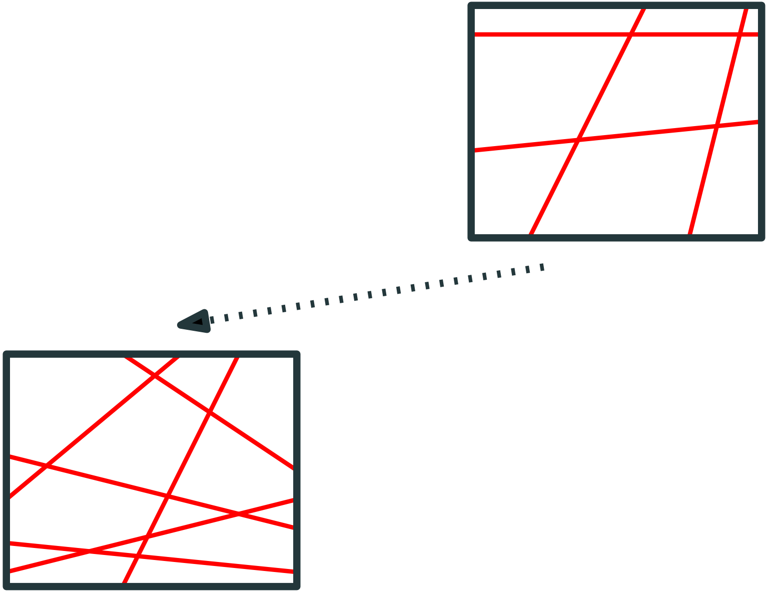

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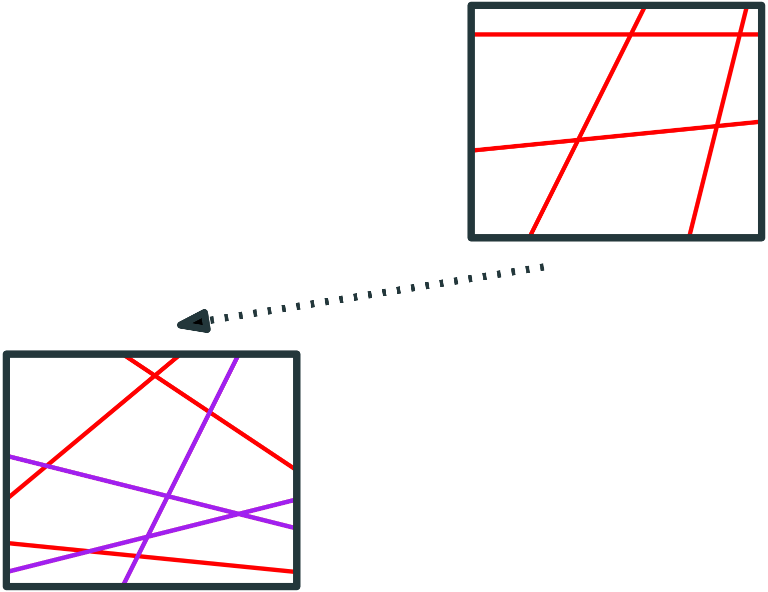

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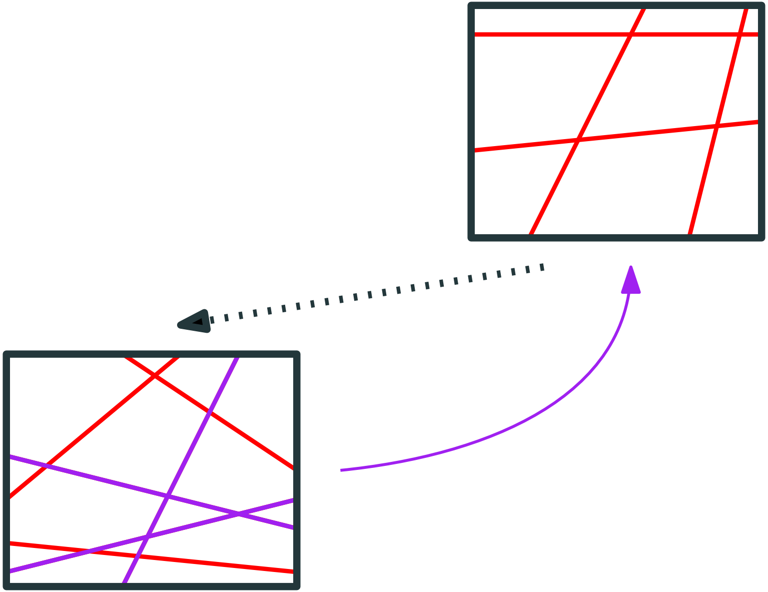

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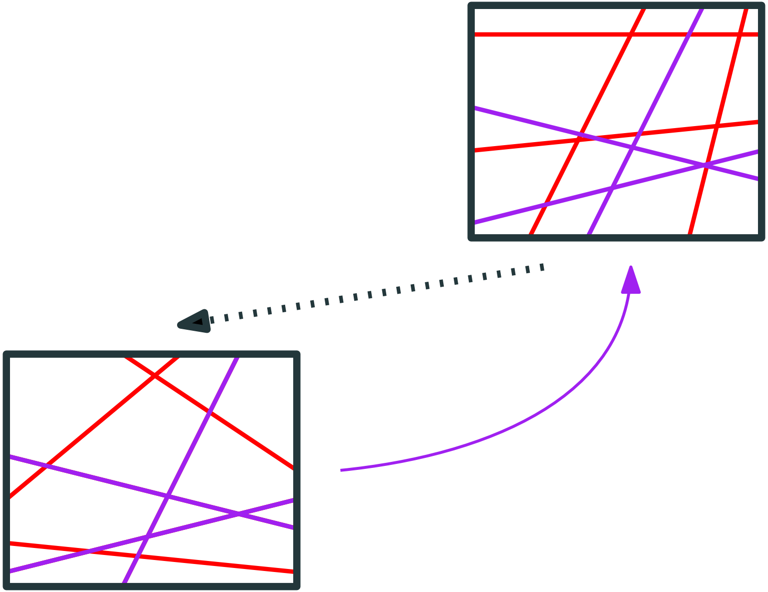

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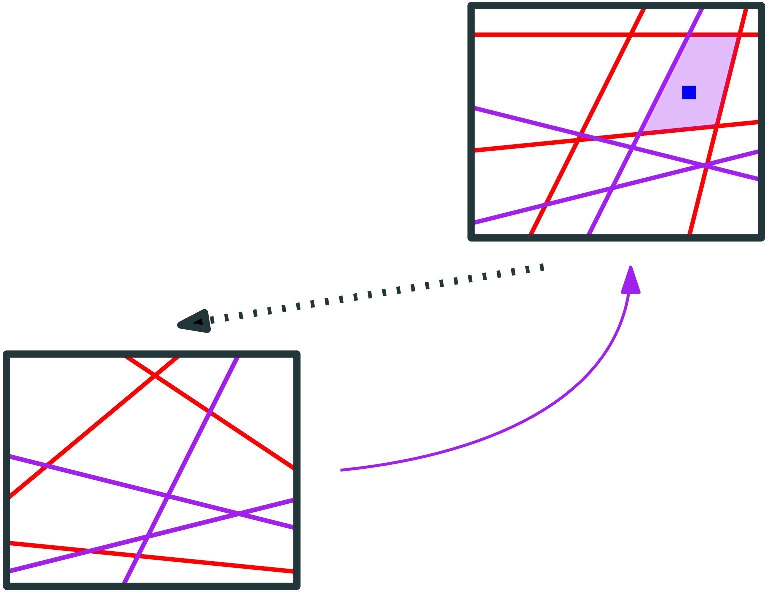

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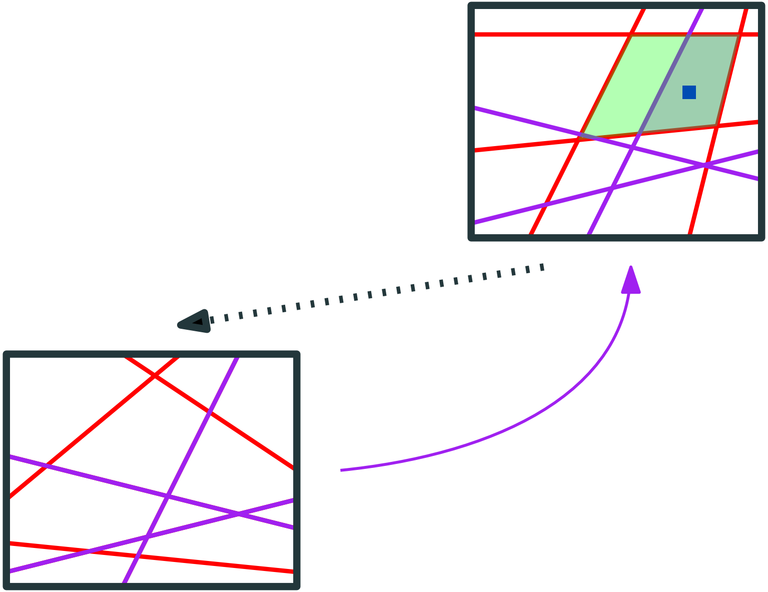

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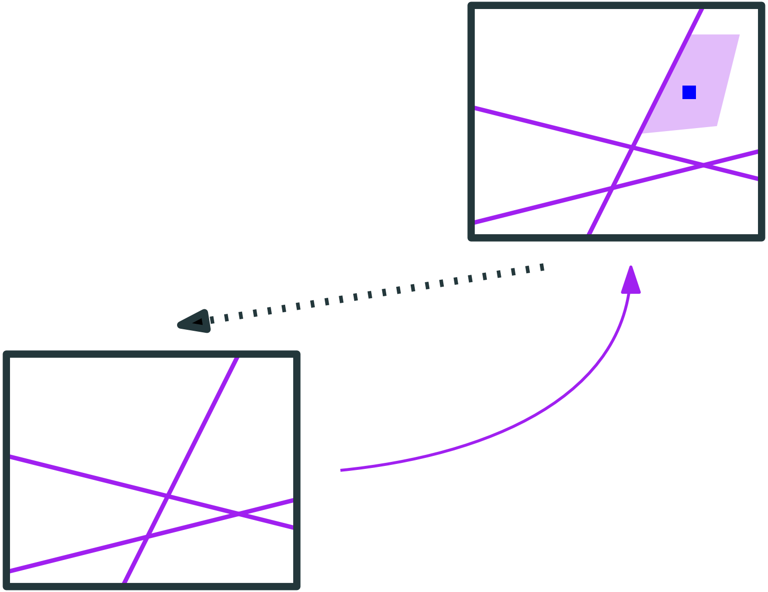

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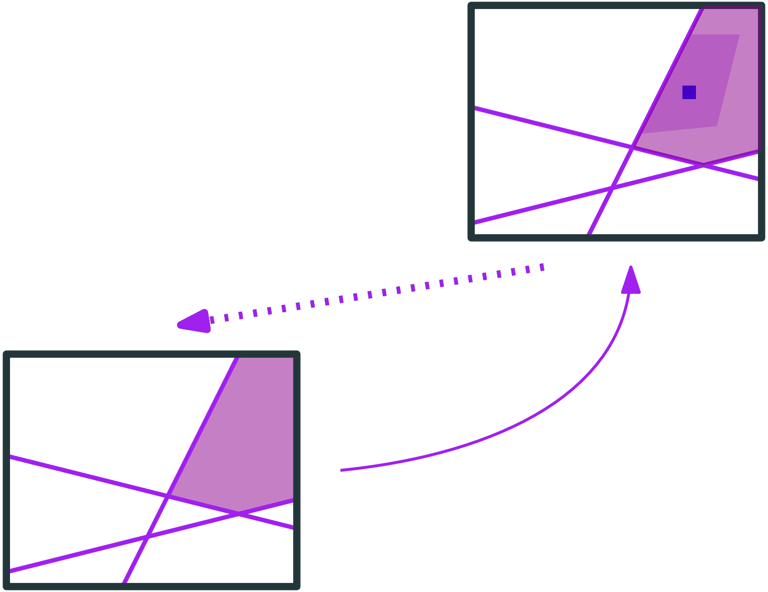

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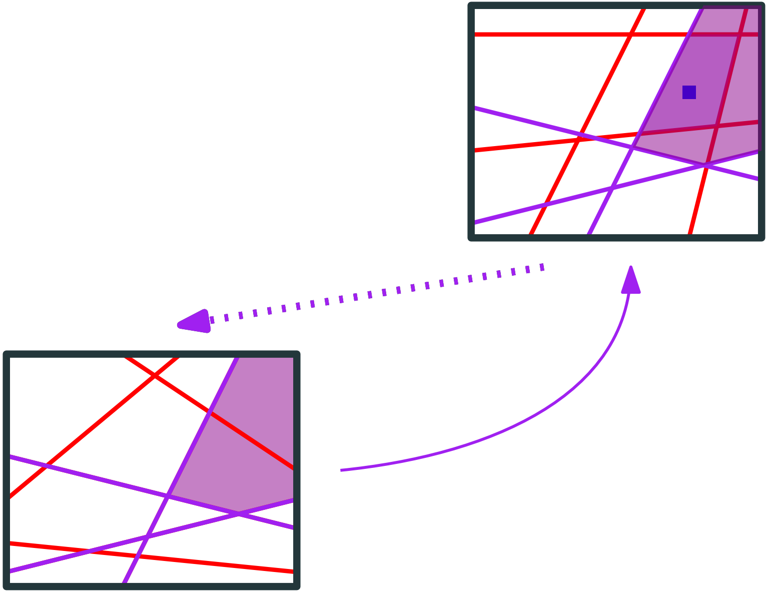

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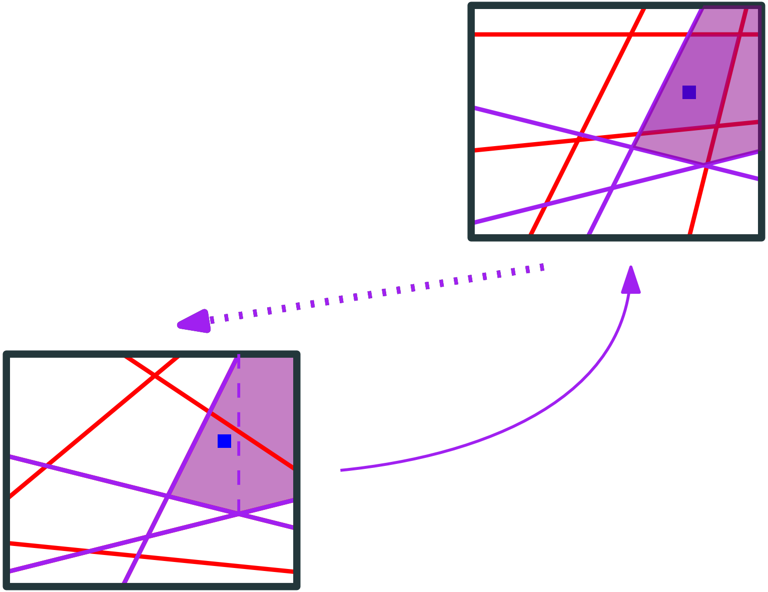

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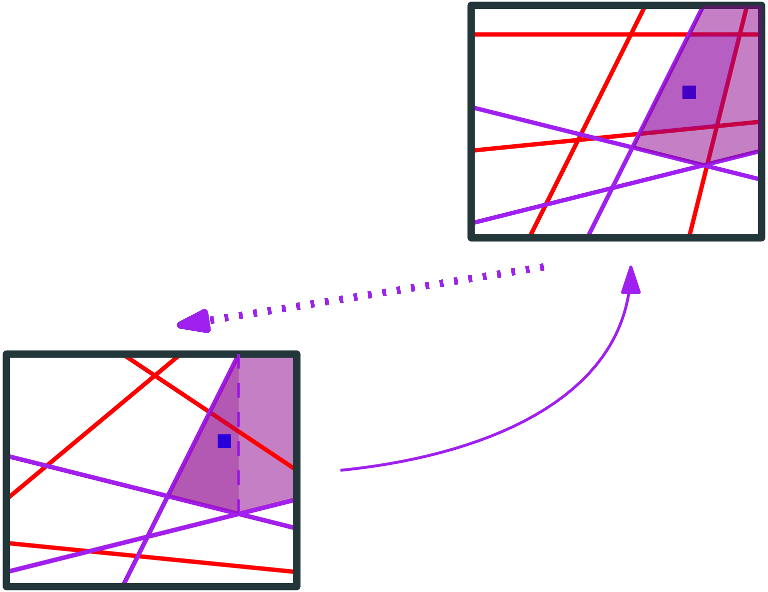

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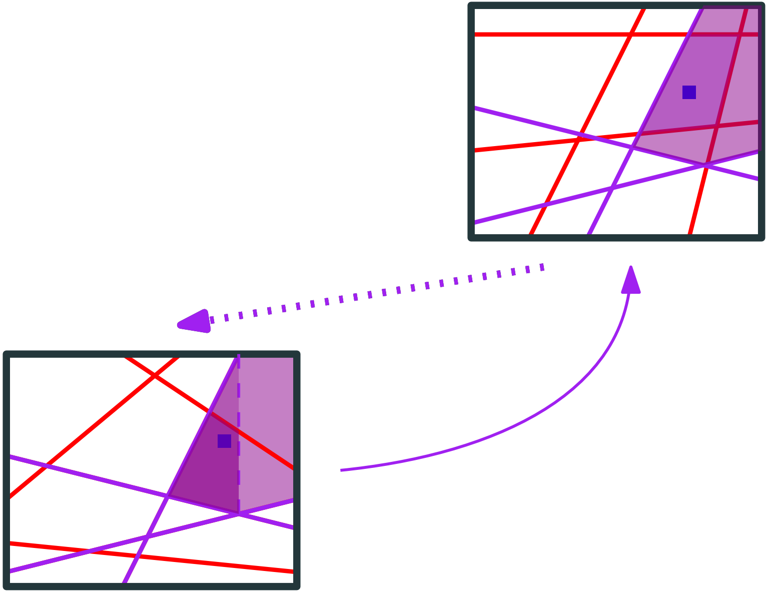

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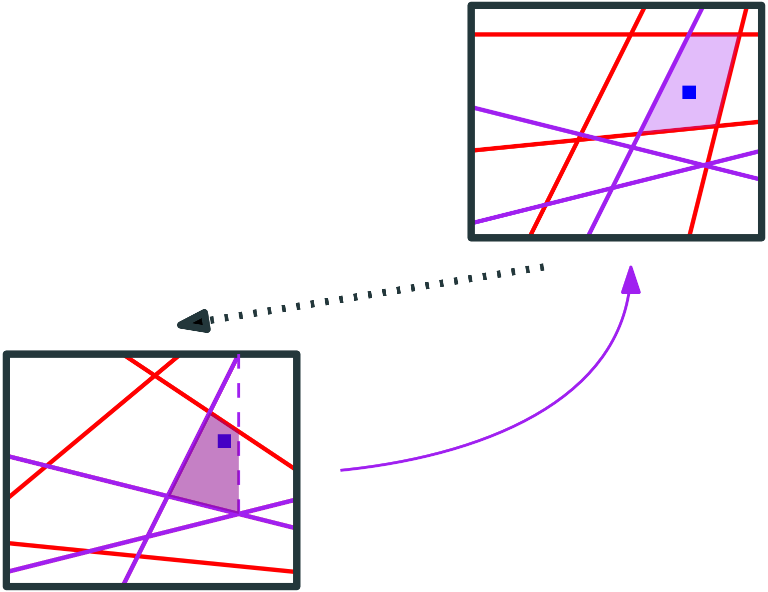

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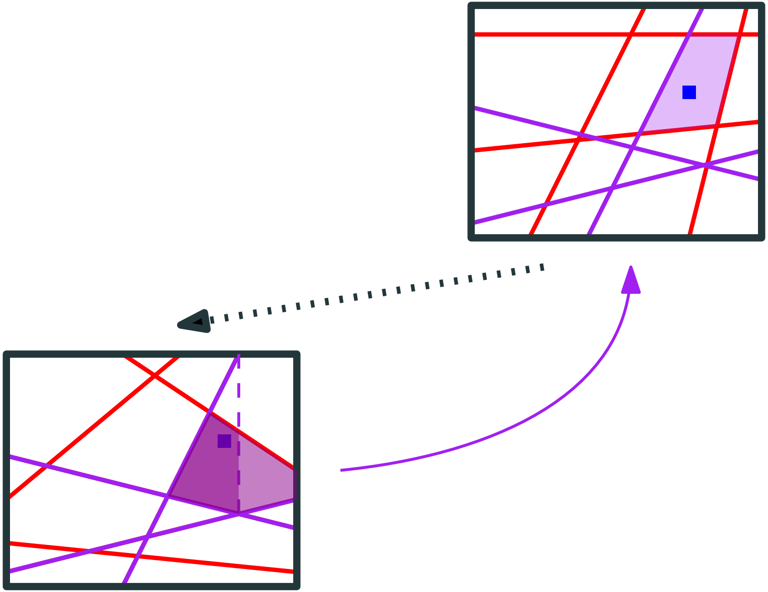
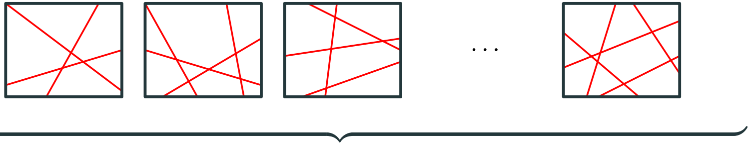


Fractional cascading of line arrangements

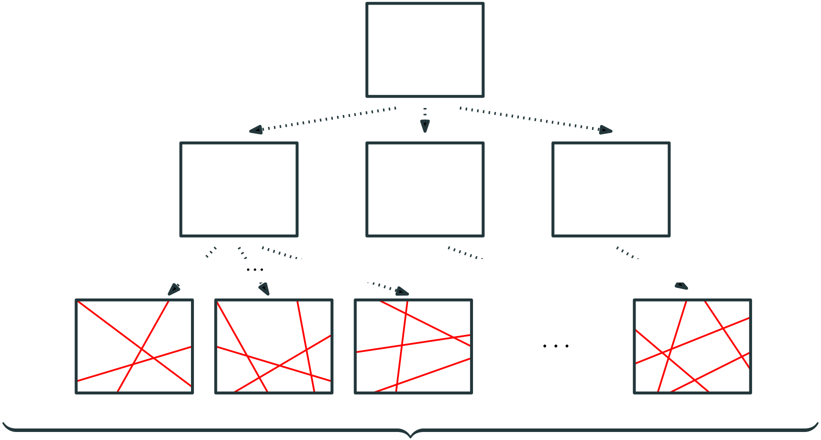
Where is our tree?



$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines and $O(n^{2/3})$ points.

Fractional cascading of line arrangements

Where is our tree? From the cutting, as they give a hierarchical tree structure!



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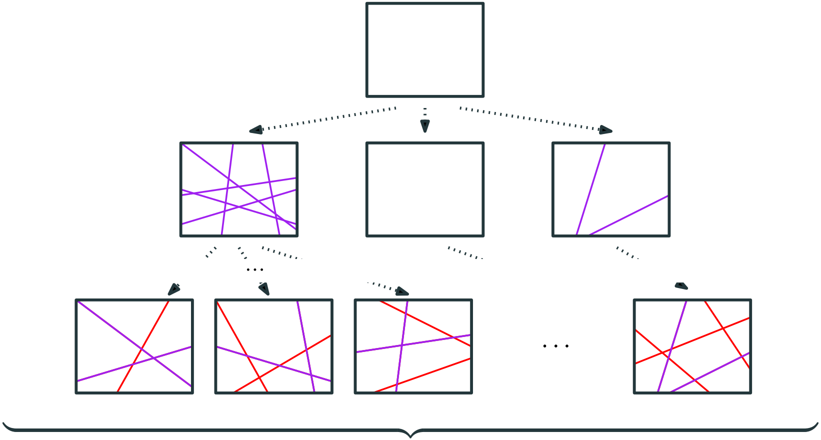

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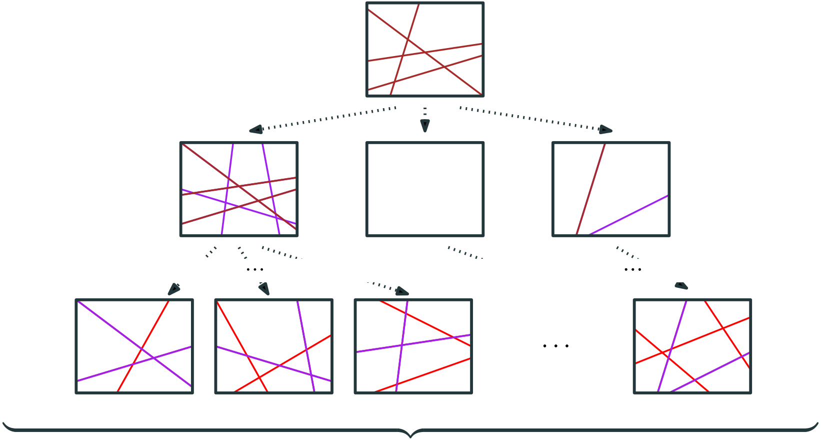

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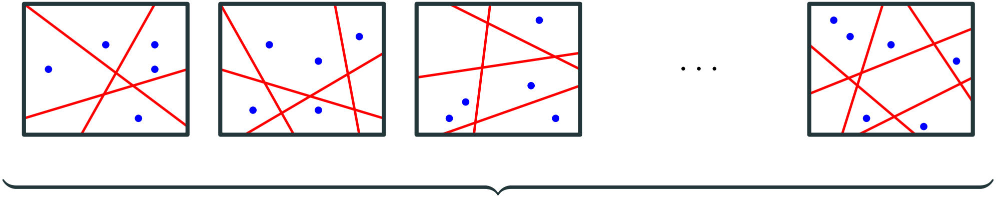
Back to Hopcroft



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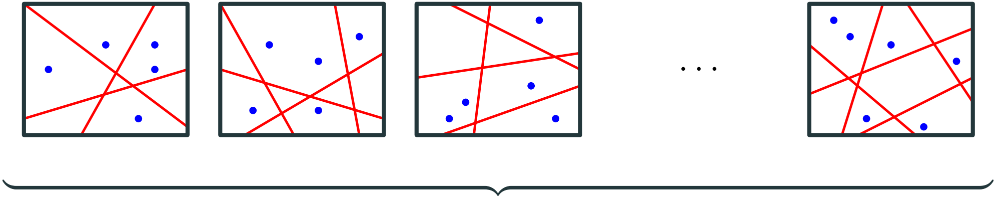
Back to Hopcroft



$O(n^{2/3})$ arrangements of $O(n^{1/3})$ lines and $O(n^{2/3})$ points.

$O(n^{4/3})$ time to do $O(n^{4/3})$ point location queries!

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Main idea: Easier to avoid logs in the decision tree model.

Outline

Low depth decision trees implies faster runtimes

Claim: If Hopcroft's problem has $O(n^{4/3})$ decision tree complexity, there exists an $O(n^{4/3})$ algorithm for Hopcroft's problem.

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This is not new, mentioned in [Matoušek, 1993], useful for 3SUM and APSP.

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Problem: Given a set $X = \{x_1, ..., x_n\}$ and a set $Y = \{y_1, ..., y_n\}$, sort the set:

$$X + Y := \{x_i + y_j \mid x_i \in X, y_j \in Y\}$$

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Theorem [Fredman, '76] Sorting $X + Y$ can be done in $O(n^2)$ comparisons.

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$O(n^4)$ such hyperplanes, can show there are $O(n^{8n})$ different cells.

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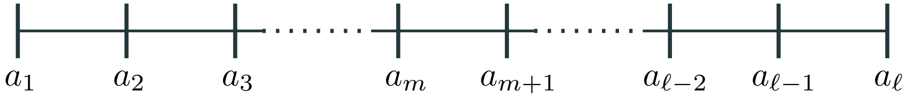

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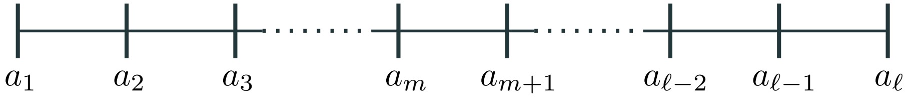

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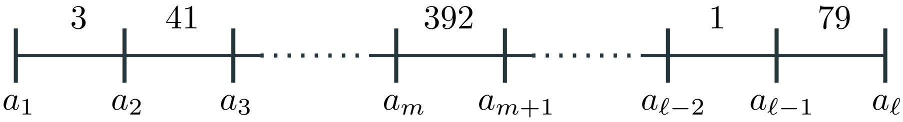

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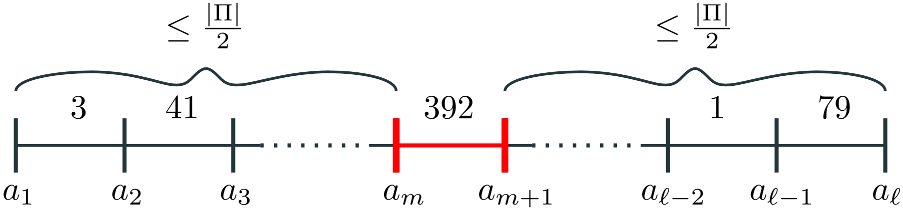

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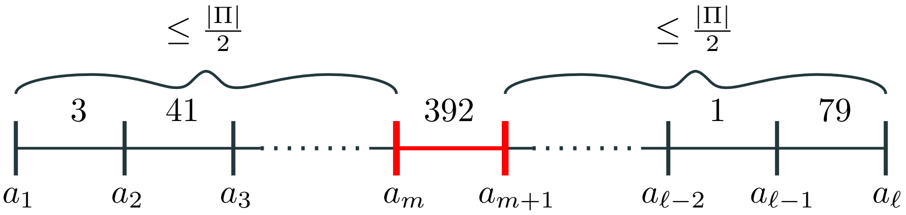

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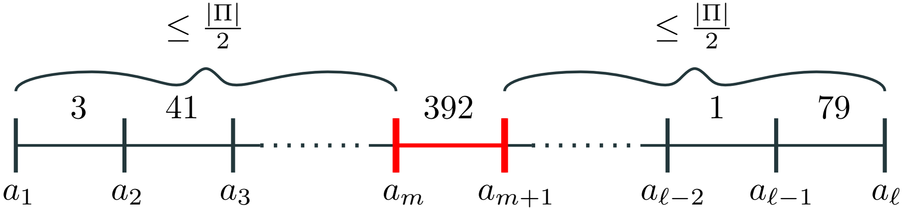

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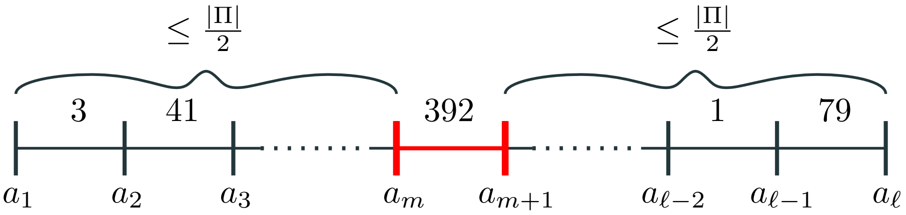

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- To find the right γ to compare with, can use hierarchical cutting tree (and use the weighted centroid).

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Approach I - Fractional Cascading

Approach II - Algebraic Decision Trees

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- Is there an analogue of our fractional cascading approach for higher dimensions?
- Are there other problems where we can improve decision tree complexity in this way and result in faster algorithms?

Thanks for listening!